Effect of quantum fluctuations in an Ising system on small-world networks

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We study quantum Ising spins placed on small-world networks. A simple model is considered in which the coupling between any given pair of spins is a nonzero constant if they are linked in the small-world network, and zero otherwise. By applying a transverse magnetic field, we have investigated the effect of quantum fluctuations. Our numerical analysis shows that the quantum fluctuations do not alter the universality class at the ferromagnetic phase transition, which is of the mean-field type. The transition temperature is reduced by the quantum fluctuations and eventually vanishes at the critical transverse field Δ_c. With increasing rewiring probability, Δ_c is shown to be enhanced.

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Various phenomena in nature and human society can be understood in terms of dynamical interaction among individual elements on a complex network. The equilibrium and dynamic properties of such systems depend strongly on the topology of the underlying network as well as how the individuals interact with each other. For example, the antiferromagnetic Ising model suffers from frustration effects on a triangular lattice, whereas it has a simple ground state on a square lattice. For the mathematical simplicity, the network is often regarded as either completely random or purely regular. In reality, however, most of the biological systems, solid-state systems, and human societies lie somewhere between these two extremes. Furthermore, it has recently been shown that such networks, now widely known as ''small-world'' networks, exhibit mixed properties. Some of which are common to completely random networks, some common to purely regular networks, and others unique to small-world networks [1,2].

A simple yet inspiring mathematical model for a small-world network has been proposed by Watts and Strogatz (WS) [3]. One starts from a regular network and ''rewires'' the links with probability p. As p varies from 0 to 1, the resulting network ''interpolates'' from a purely regular network to a completely random network. The value of the model so generated comes from the fact that it captures important physics in a wide range of physical systems, which can be described by neither a regular nor a random network.

In recent years, many authors have studied scaling properties, crossover behavior, percolation behavior [4,5], the spread of infectious diseases [6,7], signal-propagation speed [3], computational power, and synchronization [8,9] of small-world networks. More recently, phase transitions of the Ising model [10–12] and XY model [13] on small-world networks have also been studied, where the crossover from one-dimensional to mean-field behavior has been found. Although most of the works mentioned above are concerned about classical statistical problems on small-world networks, Zhu and Xiong [14,15] have recently studied the localization-delocalization transition of electronic states on a small-world network which allows a quantum mechanical hopping among sites.

In this work, we adopt a transverse-field Ising model on small-world networks, to investigate effects of the interplay between the unique topology of small-world networks and the quantum fluctuations. From the previous works [10–13], it is known that the small-world network topology enhances the correlation between the spins on the network, and leads to the mean-field behavior of the system. Since the quantum fluctuations, introduced by the transverse field in our model, tend to destroy correlations, one can anticipate a nontrivial competition between the small-world network topology and the quantum fluctuations.

Another motivation for studying the transverse-field Ising model on small-world networks is provided by the recent wide interest in quantum computing. A quantum computer can be regarded as controllable quantum spins (quantum mechanical two-state systems) interacting with each other through a network [16]. In realistic circumstances, the control of the spins, which can be achieved by carefully tuning the local magnetic field at each node and the coupling between each pair of spins, is imperfect. The effects of imperfections on quantum computing have been studied on a completely random network and have been shown to give rise to computation errors, which grow fast (exponentially or polynomially) with the number of quantum bits (or in short, qubits) [17]. It is therefore much worth studying the imperfection effects in terms of qubits on small-world networks. With the transverse field assumed uniform over the whole network, the model in our work may not directly represent a realistic quantum computer with imperfect controls, yet our work might provide a stimulation for studies in this direction.

We consider N interacting spins in a uniform transverse magnetic field Δ. We assume that the interaction between spins is Ising type. The Hamiltonian for the system is then given by

$$H = H_z + H_x = -\sum_{i<j} J_{ij}\sigma_i^z\sigma_j^z - \Delta\sum_{i=1}^{N}\sigma_i^x, \qquad (1)$$

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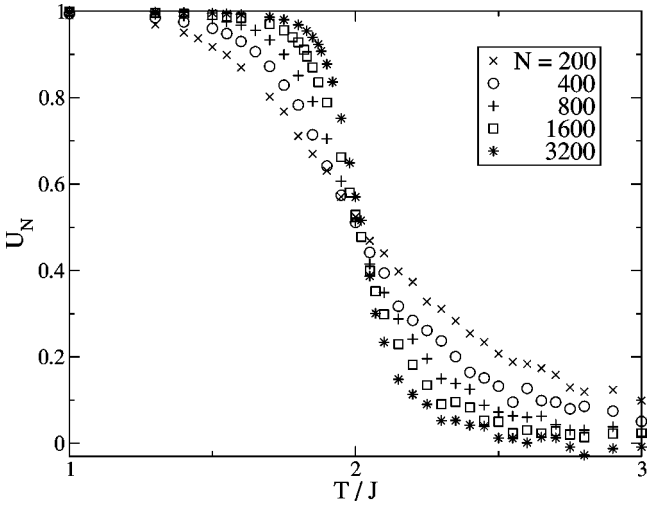

FIG. 1. The fourth order Binder cumulant U_N is drawn as a function of temperature for five different system sizes N. The parameters are given by $k=2$, $p=0.1$, and $\Delta/J=2$. In the time direction, the size was fixed at $M=30$. As N grows, the crossing point converges to one single point, which is $T/J=2.04$ in this graph.

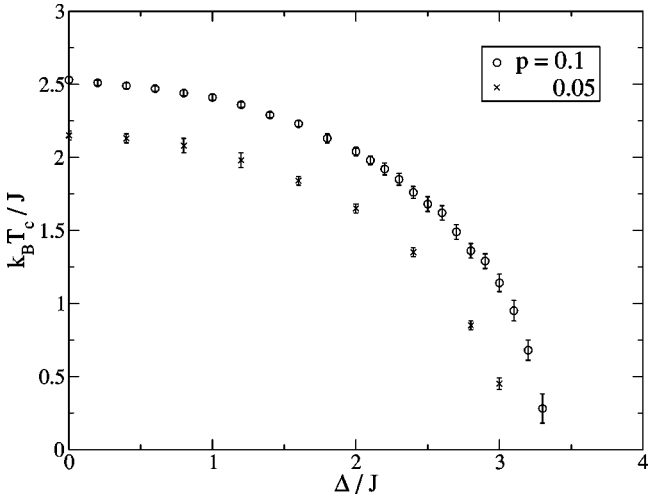

FIG. 2. Phase boundary of the transverse-field Ising model on a small-world network. The small-world network was constructed following Watts and Strogatz starting from the one-dimensional regular network with $k=2$, and rewiring the links with the probability $p=0.1$ and 0.05.

where σ^x, σ^y, and σ^z are the Pauli matrices. The coupling J_{ij} between the two spins at the ith and jth sites are defined on a small-world network (see below for the precise way the network is constructed). Namely, we regard that the spins are placed on the small-world network, and that $J_{ij}=J$ if i and j are linked on the network and $J_{ij}=0$ otherwise [18]. We will focus on the ferromagnetic case ($J>0$) in this work. For an antiferromagnetic interaction, more complications may arise due to frustration.

The small-world network is constructed in a very similar way as used by WS. We take a one-dimensional regular network with $2k$ nearest neighbors [19] with a periodic boundary condition ($\sigma^l_{i+N}\equiv\sigma^l_i, l=x,y,z$). Among the total of Nk links, we randomly choose pNk of them. One end of each chosen link is then rewired to a random site keeping the other end as a pivot point. Among the resulting networks, we discard those with disconnected parts, i.e., we only consider a network where all the nodes belong to a single cluster.

Without going into details, several properties of this model may be qualitatively understood as follows. In the classical case ($\Delta=0$), this model is known to undergo a mean-field type ferromagnetic phase transition at a finite temperature for an arbitrarily small $p>0$ [10]. Finite transverse magnetic field introduces quantum fluctuations, which compete with the correlations enhanced by the topology of the small-world network. As a consequence, one expects Δ to suppress the transition temperature T_c. In the fully quantum mechanical case ($T=0$) on the nearest-neighbor regular network ($k=1$), the Hamiltonian in Eq. (1) shows a quantum phase transition at $\Delta_c=J$ [20], which belongs to the universality class of the two-dimensional classical Ising model on an anisotropic square lattice. For the regular network with $k=2$, from which our small-world network is derived, a larger value of Δ_c is expected.

Below we will calculate the phase boundary, i.e., the transition temperature T_c as a function of Δ, using the quantum Monte Carlo simulation [21]. The universality class to which the phase transition belongs will also be determined from various critical exponents. Before we give the results, we summarize the quantum Monte Carlo method specific to our model.

We divide the inverse temperature $\beta=1/k_BT$ into M pieces with spacing $\epsilon\equiv\beta/M$. The partition function $Z(\beta)=\mathrm{tr}\exp(-\beta H)$ is then approximated by the Trotter production formula

$$Z(\beta)\approx\mathrm{tr}[\exp(-\epsilon H_x)\exp(-\epsilon H_z)]^M, \tag{2}$$

the error of which is of the order of ϵ^2 ($M\to\infty$). We now use the completeness relation

$$\sum_i \sum_{S_{i,t}=\pm 1} |S_{i,t}\rangle\langle S_{i,t}| = 1, \tag{3}$$

where $|S_{i,t}\rangle$ is the eigenstates of the σ^z_i, and put it between the tth and $(t+1)$th temperature slices in Eq. (2). We may then write [22]

$$Z(\beta)=\sum_{\{S_{i,t}\}}\exp\left[\sum_{i<j}\sum_{t=1}^{M}\epsilon J_{ij}S_{i,t}S_{j,t} + A\sum_{i=1}^{N}\sum_{t=1}^{M}S_{i,t}S_{i,t+1}+NMB\right], \tag{4}$$

where

$$A=\frac{1}{2}\ln\coth(\epsilon\Delta), \tag{5}$$

$$B=\frac{1}{2}\ln[\cosh(\epsilon\Delta)\sinh(\epsilon\Delta)]. \tag{6}$$

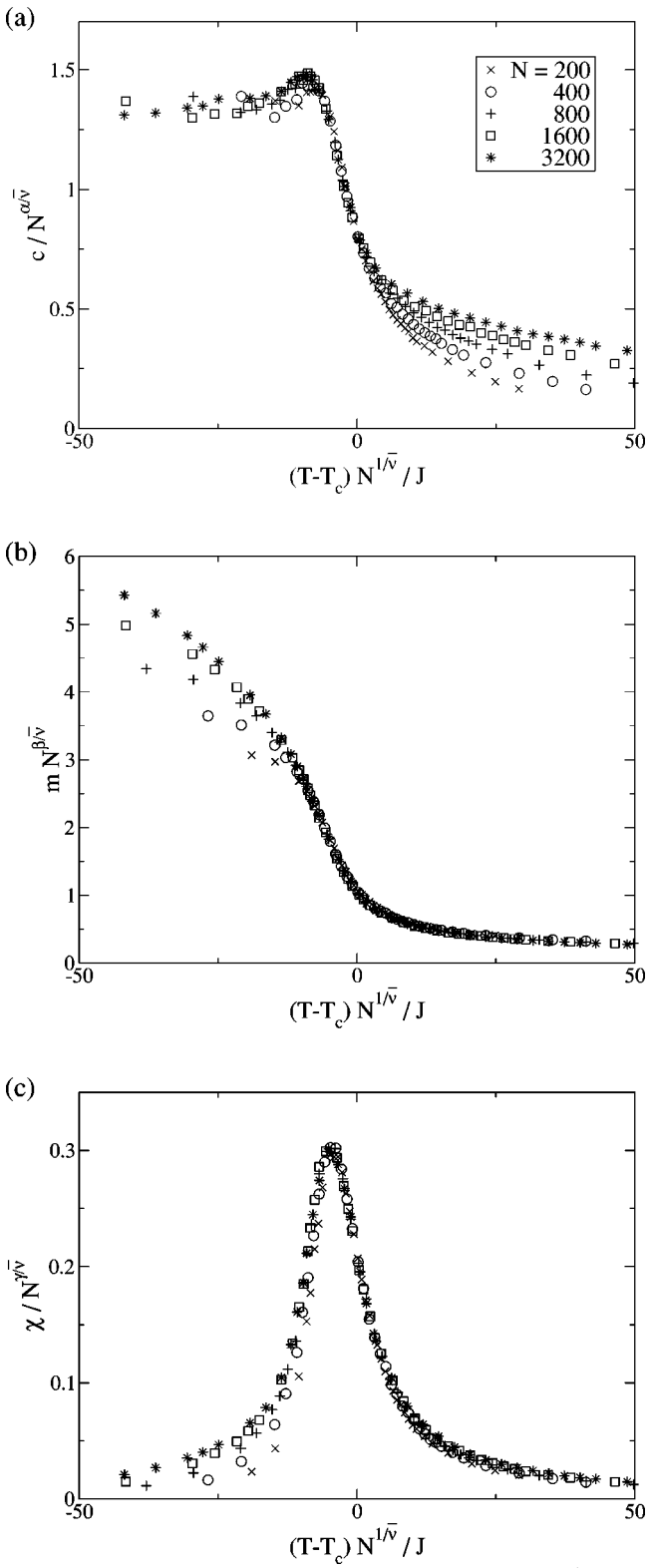

FIG. 3. Universal scaling functions at $k=2$, $p=0.1$, $\Delta/J=2$, $T_c/J=2.04$, and $M=30$: (a) specific heat, (b) magnetization, and (c) susceptibility. Each quantity is measured per site. The legend in (a) is common to all three figures.

Interpreting the temperature slices as imaginary time intervals, the above equation becomes a classical Ising model defined on a $(1+1)$-dimensional lattice. The coupling in the spatial direction (indicated by the indexes i and j) is given by J_{ij}, and contains the topology of the small-world network. The coupling in the temporal direction (indicated by the index t), on the other hand, is a nearest-neighbor coupling.

TABLE I. Critical exponents near the ferromagnetic phase transitions for $\Delta<\Delta_c$ in the transverse-field Ising model on a small-world network. For a comparison, mean-field (MF) values are also provided.

Quantity	Critical behavior	Our result	MF value
Specific heat	$c\propto\lvert T-T_c\rvert^{-\alpha}$	$\alpha=0.00\pm0.02$	$\alpha=0$
Magnetization	$m\propto(T_c-T)^{\beta}$	$\beta=0.5\pm0.1$	$\beta=1/2$
Susceptibility	$\chi\propto\lvert T-T_c\rvert^{-\gamma}$	$\gamma=1.00\pm0.02$	$\gamma=1$
Coherence Number	$N_c\propto\lvert T-T_c\rvert^{-\bar{\nu}}$	$\bar{\nu}=2.00\pm0.02$	$\bar{\nu}=2$

Due to randomness in the topology of the network, this problem is naturally suited to a numerical analysis. We have thus performed a quantum Monte Carlo analysis using the above partition function. In order to increase efficiency near and below the ferromagnetic transition temperature, a cluster algorithm was adopted [23]. Since our model is different from a usual regular Ising model, we have developed a new cluster algorithm that takes into account the small-world network topology and the anisotropy between space and time axes.

The transition temperature $T_c(\Delta)$ has been determined from the finite size scaling method. By varying temperature, we computed the fourth order Binder cumulant [24]

$$U_N(T)=\frac{1}{2}\left(3-\frac{[\langle m^4\rangle]}{[\langle m^2\rangle^2]}\right) \tag{7}$$

for several different values of N. This quantity varies from one to zero, while temperature is swept from zero (maximum order) to infinity (maximum disorder). The two different shapes of brackets, $\langle\cdots\rangle$ and $[\cdots]$, denote the thermal average and the average over rewiring configurations, respectively. A typical result is shown in Fig. 1. For large N, there is a single crossing point T_c. It is noteworthy that the result is independent of the number of time slices M once it exceeds a finite lower limit M_c, indicating that the correlation length ξ_T in the time direction is a finite fraction of β. When $T/J\gtrsim1$, the value of M_c is typically ~30 and it increases at lower temperatures.

Figure 2 shows phase diagrams in T-Δ space for $k=2$, and two different rewiring probabilities $p=0.1$ and 0.05. First of all, we find $T_c>0$ in the absence of the transverse field, which agrees with previous results [10–12]. When a small Δ is turned on, T_c remains finite although it decreases with increasing Δ. At a fixed Δ, we find that T_c increases with p, just as in the no field case.

Extrapolating the phase transition line to $T=0$, one may obtain the quantum critical transverse field Δ_c. This is an

extension of the quantum critical point of the regular Ising model [20]. From Fig. 2, one can clearly see that Δ_c increases with p. This implies that the more the Ising system is rewired, the more resilient it is to quantum fluctuations.

We now characterize the ferromagnetic phase transition by determining the universality class to which it belongs. In the pure classical case ($\Delta=0$), it is known that the phase transition is mean-field-like [10,11,13]. To address the question whether this is still the case for finite values of the transverse field Δ, we investigated the scaling behavior of the phase transition described above. First of all, the Binder cumulant was fitted to a scaling function of the form

$$U_N(T)=\tilde{U}((T-T_c)N^{1/\bar{\nu}}). \tag{8}$$

Due to the infinite-range nature of the small-world network, the above exponent $\bar{\nu}$ describes the divergence of coherence number N_c instead of the correlation length ξ [13,25]. More explicitly, we may write

$$N_c \propto |T-T_c|^{-\bar{\nu}}, \tag{9}$$

near the transition.

The other critical exponents have also been obtained from various physical quantities by fitting them to scaling functions. For example, the specific heat per spin is fitted to

$$c(T)=N^{\alpha/\bar{\nu}}\tilde{c}((T-T_c)N^{1/\bar{\nu}}). \tag{10}$$

Figure 3 shows an example of the scaling functions for (a) specific heat, (b) magnetization, and (c) susceptibility per spin. If we choose appropriate exponents, results from systems of different sizes clearly collapse to one single curve near T_c. As T moves away from the scaling regime, the curves deviate. The best fitting values of the exponents are summarized in Table I. It turned out that they are the same as those of the mean-field transition to a very high precision. Therefore, we conclude that the quantum fluctuations introduced by the transverse field do not alter the universality class of the ferromagnetic phase transition in the Ising models on small-world networks.

So far, we have studied the scaling properties of an Ising system on small-world networks at finite temperatures. However, the quantum critical point Δ_c at $T=0$ is in itself of much interest, since in general the universality class of a quantum critical point is different from that of classical transitions at finite temperatures [20]. Whether the small-world network will change the universality class of the quantum critical point as compared to that in the regular Ising model is a highly intriguing question. In order to address this question, however, one has to use a different technique than those used above, because the results from the quantum Monte Carlo simulations become unreliable near $T=0$. Therefore, we leave it as a topic for further study.

In summary, we have used the quantum Monte Carlo simulations to obtain the phase diagram of an Ising system on small-world networks in the presence of a transverse magnetic field. The ferromagnetic phase persisted at finite Δ, although the effect of quantum fluctuations introduced by the transverse field was manifested by the decrease of T_c. At a fixed Δ, we have also shown that T_c increases with the rewiring probability p. From various scaling exponents, we have argued that the ferromagnetic phase transition at finite field was still of a mean-field type. Eventually, T_c decreases to zero at a quantum critical point at a finite field Δ_c, but Δ_c increases with increasing p. Since the ferromagnetic region increases with p in both T and Δ directions, we conclude that the small-world topology of the spin system competes against both thermal and quantum fluctuations, and enhances the correlation and ordering of the spins.

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