Local and nonlocal contents in *N*-qubit generalized Greenberger-Horne-Zeilinger states

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We investigate local contents in *N*-qubit generalized Greenberger-Horne-Zeilinger (GHZ) states. We suggest a decomposition for correlations in the GHZ states into a nonlocal and fully local part, and find a lower and upper bound on the local content. Our lower bound reproduces the previous result for $N = 2$ [V. Scarani, Phys. Rev. A **77**, 042112 (2008)] and decreases rapidly with N.

I. INTRODUCTION

Bell's theorem has revealed that local variable theories cannot reproduce all statistical predictions of quantum theory, and it highlights the statistical incompatibility between classical local variable theories and quantum theory [1,2]. When the Bell-type inequalities are violated, nonlocality appears. However, even if the observations on a given system of particle pairs exhibit nonlocality, it does not necessarily imply that all individual pairs in the system behave nonlocally. It may be possible that some fraction of the pairs behave nonlocally, while the other pairs behave locally.

This issue was investigated carefully first by Elitzur, Popescu, and Rohrlich (their approach is referred to here as EPR2) [3] in terms of the local contents in a given nonlocal correlation. Since then, several authors have generalized and further discussed the idea [4–9]. For example, the EPR2 approach has been related to another noticeable question, the simulation of quantum correlations with other resources, which has been proved useful in the task of simulating entanglement [4]. Barrett *et al.* [6] gave an upper bound of the weight of the local component in a $d \times d$ system. Scarani [5] presented an improved lower and upper bound of the local content in the family of pure two-qubit states and the first example of a lower bound of the local content in a pure two-qutrit system. Later, Zhang *et al.* [7] extended this lower bound to the mixed two-qubit states. In [9], a new EPR2 decomposition has been given, which can make their local content reach the upper bound in a wide range of two-qubit pure states.

In this paper, we investigate local contents in *N*-qubit generalized Greenberger-Horne-Zeilinger (GHZ) states. In Sec. II, we first examine general requirements for optimization of the weight of the local part in a convex decomposition of a given quantum distribution into local and nonlocal parts. Guided by the requirements, we suggest local probability distributions for *N*-qubit GHZ states, which give the lower bound of the local contents of such states, in Sec. III. Our result for $N = 2$ reproduces the lower bound in [5]. It is also noted that the lower bound on the local content decreases rapidly with N. In Sec. IV, we briefly discuss the upper bound based on Bell-type inequalities.

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II. EPR2 APPROACH

Before we go further, here we first review the notion of *local content* suggested first by Elitzur, Popescu, and Rohrlich [3]. We follow the conventions in Ref. [5].

Consider a system of *N* parties, labeled by $1,2,\ldots,N$. On each party j, one measures any observable A_j in a given set $\mathcal{A}_j$. The measurement output of A_j is denoted by r_j. The joint probability distribution for measurements on the system is denoted by $P(r_1,r_2,\ldots,r_N|A_1,A_2,\ldots,A_N)$. If the parties are noncommunicating but share classical information, the joint probability distribution takes the following form:

$$P(r_1,r_2,\ldots,r_N|A_1,A_2,\ldots,A_N)$$
$$= \int d\mu(\lambda)P(r_1|A_1,\lambda)P(r_2 \mid A_2,\lambda)\ldots P(r_N|A_N,\lambda), \quad (1)$$

where $\lambda \in \Lambda$ denotes the collective local hidden variables that represent the shared classical information and Λ is the space of all hidden variables. The form of the distribution in (1) leads to a set of constraints on the joint distributions (Bell-type inequalities) for any fixed number of measurements on each party. If there exist joint probability distributions that violate the inequalities, they would not be written as (1) and are thus nonlocal.

The quantum correlations are obtained by general measurements on quantum states, and the joint probability distribution is given by

$$P_Q(r_1,r_2,\ldots r_N|A_1,A_2,\ldots,A_N;\rho)$$
$$= \text{Tr}\big(\Pi_{r_1}^{A_1} \otimes \Pi_{r_2}^{A_2} \otimes \cdots \otimes \Pi_{r_N}^{A_N}\rho\big). \quad (2)$$

Here ρ is the density matrix for a quantum state of the system of *N* parties. $\Pi_{r_j}^{A_j}$ is the projector on the subspace associated to the measurement result r_j of the observable A_j performing on party j. There exist quantum probability distributions that are not local, as proved by Bell [1].

The EPR2 approach is a quantitative notion of nonlocality [3]. The main idea is to consider the possible decomposition of P_Q into a local part P_L and a nonlocal part P_{NL}:

$$P_Q = w(\rho)P_L + [1 - w(\rho)]P_{NL}, \quad (3)$$

where the weight $w \in [0,1]$ of the local component is required to be independent of the measurements and the outcomes. Obviously, the convex combination (3) is not unique. The point is to find the local part P_L that maximizes the weight w. The resulting optimal value w_{opt} of w is defined as the

local content in the joint probability distribution P_Q. The local content $w_{opt}(\rho)$ should be 1 if ρ is a product state, and 0 if ρ is a maximally entangled state [3,6].

The full optimization of the local part P_L is highly nontrivial. Several authors have investigated the local contents in two-qubit and two-qutrit states, and proposed upper and lower bounds on the local contents [3,5–7,9]. Here we intend to give lower and upper bounds of the local content w_{opt} in the N-qubit generalized GHZ states of the form

$$|\Psi_n(\alpha)\rangle = \cos\alpha|0\ldots0\rangle + \sin\alpha|1\ldots1\rangle, \qquad (4)$$

where $\alpha \in [0, \pi/4]$.

III. LOWER BOUND ON THE LOCAL CONTENT

Here we provide the lower bound $w_{opt}^<$ of the local content w_{opt} by finding reasonable local probability distribution function P_L guided by the following requirements: (i) As the the nonlocal part P_{NL} is a probability distribution and nonnegative,

$$P_Q(r_1, r_2, \ldots r_N | A_1, A_2, \ldots, A_N; \rho)$$
$$\geq w P_L(r_1, r_2, \ldots, r_N | A_1, A_2, \ldots, A_N) \qquad (5)$$

for all possible local measurements A_j and outcomes r_j. In particular, P_L should be zero whenever P_Q is zero. (ii) As $P_L(r_1, r_2, \ldots, r_N | A_1, A_2, \ldots, A_N)$ is a real probability distribution,

$$\sum_{r_1, r_2, \ldots, r_N} P_L(r_1, r_2, \ldots, r_N | A_1, A_2, \ldots, A_N) = 1. \qquad (6)$$

For an arbitrary N-qubit state ρ_N, the joint probability distribution is given by

$$P_Q(r_1, \ldots, r_N | A_1, \ldots, A_N; \rho_N) = \text{Tr}\left(\Pi_{r_1}^{A_1} \otimes \cdots \otimes \Pi_{r_N}^{A_N} \rho_N\right) \qquad (7)$$

with the the projectors defined by

$$\Pi_{r_j}^{A_j} = \tfrac{1}{2}(I + r_i \vec{n}_i \cdot \vec{\sigma}), \qquad (8)$$

where $r_1, r_2 = \pm 1$, $\vec{\sigma}$ denotes the three Pauli matrices, and $\vec{n}_i = (\sin\theta_i \cos\varphi_i, \sin\theta_i \sin\varphi_i, \cos\theta_i)$. Without loss of generality, by readjusting the quantization axis if necessary [10], we assume that

$$\varphi_1 + \cdots + \varphi_N = \pi. \qquad (9)$$

Then the quantum joint probability distribution corresponding to the N-qubit GHZ state in Eq. (4) can be written as

$$P_Q = \frac{\cos^2\alpha}{2^N}\prod_{j=1}^{N}(1 + r_j\cos\theta_j) + \frac{\sin^2\alpha}{2^N}\prod_{j=1}^{N}(1 - r_j\cos\theta_j)$$
$$- \frac{\sin(2\alpha)}{2^N}\prod_{j=1}^{N} r_j\sin\theta_j. \qquad (10)$$

A. $N = 2$ case

The original EPR2 paper [3] proposed an explicit local probability distribution P_L, which leads to a decomposition of the form in (3) with $w(\alpha) = [1 - \sin(2\alpha)]/4$. This is the first known lower bound on $w_{opt}(\alpha)$. They proved that the bound is

tight for the maximally entangled state and under a reasonable continuity assumption; the singlet state of two qubits is fully nonlocal. However, for the product state which is fully local, w equals 1/4 instead of 1. So this decomposition is not optimal. Later, Scarani suggested a modified explicit local probability distribution P_L, which can lead to an EPR2 decomposition with $w_{opt}^<(\alpha) = 1 - \sin(2\alpha)$ [5].

Here we exploit a method to find local distribution function P_L, which can be easily extended to the cases with $N > 2$. In order to optimize w in the decomposition (3) as much as possible, we take a note of the requirement as discussed in Sec. II that $P_L = 0$ whenever $P_Q = 0$ and that P_L should approach P_Q as much as possible. In the special case of $r_1 = r_2 = 1$ and $\theta_1 = \theta_2 = \theta$, the quantum probability distribution in (10) reduces to

$$P_Q = \tfrac{1}{4}\{2\cos\theta[1 + \cos(2\alpha)] - \sin^2\theta[1 + \sin(2\alpha)]\}. \qquad (11)$$

We note that $P_Q = 0$ only when $\theta = \theta_0$, where

$$\cos\theta_0 \equiv -\frac{1 - \tan\alpha}{1 + \tan\alpha}. \qquad (12)$$

That means that we must have $P_L = 0$ at $\theta = \theta_0$. Besides, P_L should approach P_Q as close as possible. We thus suggest a local probability distribution P_L of the form

$$P_L = \frac{1}{4}\left[1 + \text{sgn}(\cos\theta_1)\min\left(1, \left|\frac{\cos\theta_1}{\cos\theta_0}\right|\right)\right]$$
$$\times \left[1 + \text{sgn}(\cos\theta_2)\min\left(1, \left|\frac{\cos\theta_2}{\cos\theta_0}\right|\right)\right]. \qquad (13)$$

It is easy to see that this form can ensure that $P_L = 0$ whenever $P_Q = 0$. Obviously, in the special situation that $\theta_1 = \theta_2 = \theta$, if $P_Q = 0$, then P_L is zero. In a general situation that $\theta_1 \neq \theta_2$ and $P_Q = 0$, it follows from the form in (10) that $P_Q = 0$ at (θ_1, θ_2) such that either $\cos\theta_1 > \cos\theta_0 > \cos\theta_2$ or $\cos\theta_2 > \cos\theta_0 > \cos\theta_1$. When $\cos\theta_1 > \cos\theta_0 > \cos\theta_2$, the second factor in Eq. (13) vanishes, and vice versa.

Previously, we discussed the situation that $r_1 = r_2 = 1$, but a valid local probability distribution should contain all local measurements and outcomes. So we give the complete local probability distribution P_L as

$$P_L = \frac{1}{4}\prod_{j=1}^{2}\left[1 + r_j\text{sgn}(\cos\theta_j)\min\left(1, \left|\frac{\cos\theta_j}{\cos\theta_0}\right|\right)\right]. \qquad (14)$$

Note that this form of the local distribution function is identical to the one in Ref. [5] ($\frac{1+\tan\alpha}{1-\tan\alpha} = \frac{\cos 2\alpha}{1-\sin 2\alpha}$). Once the local component P_L is fixed, the weight $w(\alpha)$ is optimized to give the lower bound $w_{opt}^<$ on the local content by minimizing the function $f(\theta)$, defined by

$$f(\theta) \equiv \frac{P_Q(\theta)}{P_L(\theta)}, \qquad (15)$$

where $P_Q(\theta)$ and $P_L(\theta)$ is the quantum and local joint probability functions, respectively, in the special case $\theta_1 = \theta_2 = \theta$. For the present case ($N = 2$),

$$f(\theta) = \frac{1 + 2\cos(2\alpha)\cos\theta + \cos^2\theta - \sin(2\alpha)\sin^2\theta}{\left[1 + \text{sgn}(\cos\theta)\min\left(1, \left|\frac{\cos\theta}{\cos\theta_0}\right|\right)\right]^2}. \qquad (16)$$
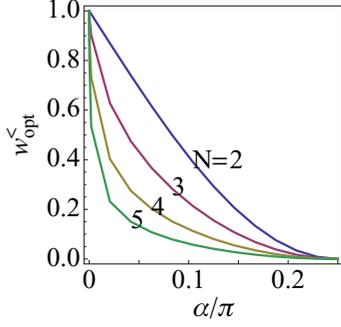
FIG. 1. (Color online) Lower bound $w_{\text{opt}}^<$ on the local content w_{opt} in pure N-qubit generalized GHZ states as a function of α for $N = 2, 3, 4, 5$ (top to bottom).

The local distribution function proposed in (14) allows the resulting lower bound $w_{\text{opt}}^<$ to reach $1 - \sin(2\alpha)$ obtained previously by Scarani [5]. The profile of the lower bound $w_{\text{opt}}^<(\alpha)$ of local content versus α is shown in Fig. 1 ($N = 2$). Clearly, $w_{\text{opt}}^<(\alpha)$ decreases with α, eventually vanishing at $\alpha = \pi/4$, as it should since the degree of entanglement in the GHZ state in Eq. (4) increases with α, reaching the maximal entanglement at $\alpha = \pi/4$.

B. $N = 3$ case

As before, we first consider the special situation $r_1 = r_2 = r_3 = 1$ and $\theta_1 = \theta_2 = \theta_3 = \theta$, where the quantum probability distribution (10) is reduced to

$$P_Q = \tfrac{1}{16}[\cos(\alpha - 3\theta/2) + 3\cos(\alpha + \theta/2)]^2. \quad (17)$$

$P_Q = 0$ only when $\theta = \theta_0$, where

$$\cos\theta_0 \equiv -\frac{1 - \tan^{\frac{2}{3}}\alpha}{1 + \tan^{\frac{2}{3}}\alpha}. \quad (18)$$

Following the same lines as in the case of $N = 2$, we suggest for the local probability distribution

$$P_L = \frac{1}{8}\prod_{j=1}^{3}\left[1 + r_j \operatorname{sgn}(\cos\theta_j)\min\left(1, \left|\frac{\cos\theta_j}{\cos\theta_0}\right|\right)\right]. \quad (19)$$

Given the form of local distribution function (19), the lower bound on the local content is again determined by minimizing the function $f(\theta)$ in Eq. (15). Note that unlike the previous case, the lower bound of local content cannot reach $1 - \sin(2\alpha)$ except for the maximally entangled state and the product state. For example, when $\alpha = \frac{\pi}{12}$, $w_{\text{opt}}^< = 0.28$. The profile of $w_{\text{opt}}^<(\alpha)$ versus α is shown in Fig. 1 ($N = 3$). As in the previous case with $N = 2$, the lower bound on local content decreases with α and vanishes at $\alpha = \pi/4$. It is interesting to note that the lower bound on local content decreases faster in this case than for $N = 2$. As we will see below for N-qubit states, this trend is general and our lower bound on local contents in the GHZ state (4) decreases rapid with N.

C. General N-qubit case

Following similar lines as above, we define θ_0 by

$$\cos\theta_0 \equiv -\frac{1 - \tan^{\frac{2}{N}}\alpha}{1 + \tan^{\frac{2}{N}}\alpha}, \quad (20)$$

at which $P_Q = 0$ only when $\theta_1 = \cdots = \theta_N = \theta_0$. We then suggest the following form of the local probability distribution P_L:

$$P_L = \frac{1}{2^N}\prod_{j=1}^{N}\left[1 + r_j \operatorname{sgn}(\cos\theta_j)\min\left(1, \left|\frac{\cos\theta_j}{\cos\theta_0}\right|\right)\right]. \quad (21)$$

It is interesting to note that the resulting lower bound $w_{\text{opt}}^<$ on the local content decreases rapidly with N. Its profile versus α for $N = 2, 3, 4, 5$ is shown in Fig. 1.

IV. UPPER BOUND ON THE LOCAL CONTENT

So far we have focused on the lower bound $w_{\text{opt}}^<(\rho)$ on the local content $w_{\text{opt}}(\rho)$. Let us now briefly discuss the upper bound $w_{\text{opt}}^>(\rho)$. As pointed out in Ref. [6], any Bell inequality can lead to an upper bound on $w_{\text{opt}}(\rho)$. Following Refs. [5,6], suppose that a Bell inequality $P \leqslant P_L^*$ with a constant P_L^* holds for all local probability distributions. Let P_{NS}^* be the maximum value of P under the nonsignaling condition. Then by Eq. (3) and the Bell inequality, $P_Q^*(\rho) \leqslant w_{\text{opt}}(\rho)P_L^* + [1 - w_{\text{opt}}(\rho)]P_{NS}^*$ where P_Q^* is the quantum value of P for the best choice of measurements. That is, the upper bound of the local content is given by

$$w_{\text{opt}}^> = \frac{P_{NS}^* - P_Q^*}{P_{NS}^* - P_L^*}. \quad (22)$$

An upper bound $w_{\text{opt}}^>$ for $N = 2$ was obtained based on this method in Ref. [5]. Here we thus focus on the case of $N \geqslant 3$, where a Bell-type inequality was derived and shown to be violated maximally by GHZ states in Ref. [11]. One can show that $I_L^* = 1$, $P_{NS}^* = 2^{N-2}$, and $P_Q^* = \sqrt{2^{N-2}\sin^2(2\alpha) + \cos^2(2\alpha)}$. It immediately follows that

$$w_{\text{opt}}^> = \frac{2^{N-2} - \sqrt{2^{N-2}\sin^2(2\alpha) + \cos^2(2\alpha)}}{2^{N-2} - 1}. \quad (23)$$

Obviously, for product states ($\alpha = 0$), the upper bound reaches 1, which is optimal. However, for the maximally entangled state ($\alpha = \pi/4$), the upper bound is not optimal, and approaches 1 as $N \to \infty$ (to be compared with the result in the bipartite case, $N = 2$, in Ref. [6]).

One can also consider the upper bound based on the Mermin-Ardehali-Belinskii-Klyshko (MABK) inequality [12] for the GHZ states (4). In this case, we find that the upper bound is 0 for the maximally entangled state. However, for the generalized GHZ states such that $\sin(2\alpha) \leqslant 1/\sqrt{2^{N-1}}$, the upper bounds based on the MABK inequality are 1 again. This is because such states do not violate MABK inequalities [13].

V. CONCLUSION

In this paper, we have provided a decomposition for correlations in N-qubit generalized GHZ states into a nonlocal and fully local part. A general form of nontrivial local

probability distribution P_L of N qubits has been proposed based on the properties of the convex decomposition of the quantum joint probability distribution into local and nonlocal parts, and thereby a lower bound on the local content in the GHZ states has been suggested. The improved local probability distribution in [5] for pure two-qubit states turns out to be a special case of our results. Moreover, for a fixed value of α, our lower bound on the local content decreases rapidly with

N. We have also investigated the upper bound on the local content based on Bell-type inequalities.

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[10] Given a choice of the quantization axes, let φ_j ($j = 1, \ldots, N$) be the azimuthal angle of the jth qubit, in general, with $\sum_j \varphi_j \neq \pi$. Then rotate the axes counterclockwise around the z axis by angle $\varphi_0 = (\sum_j \varphi_j - \pi)/N$ so that $\varphi_j \rightarrow \varphi'_j = \varphi_j - \varphi_0$, and hence $\sum_j \varphi'_j = \pi$.

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