

Research Training Workshop 2016

Quantum Control Lab

KU SUMMER 2015 INTERNSHIP PROGRAM

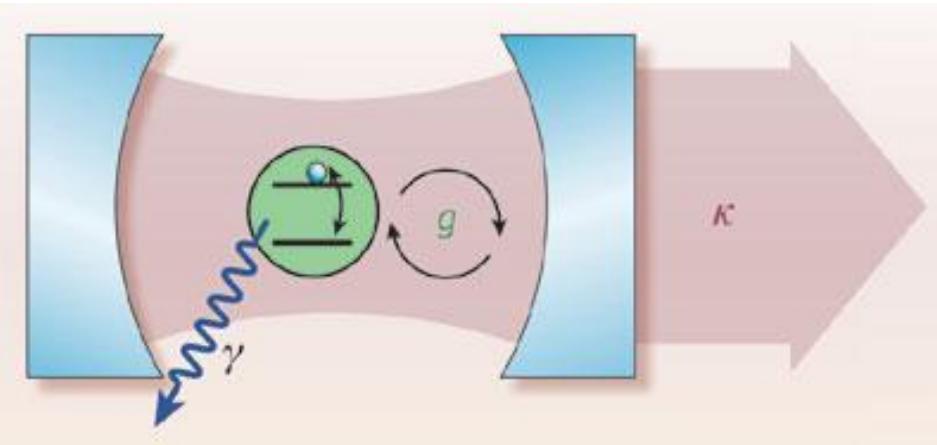
- Circuit Q.E.D
- Decoherence effects

Presenter: NGUYEN LE DUC THINH

Alexandre Blais, Physical review A 69, 062320 (2004)
K.blum, Density Matrix theory and applications (2012)
Nathan K.Langford, Circuit QED-Lecture Notes (2013)

CIRCUIT QED

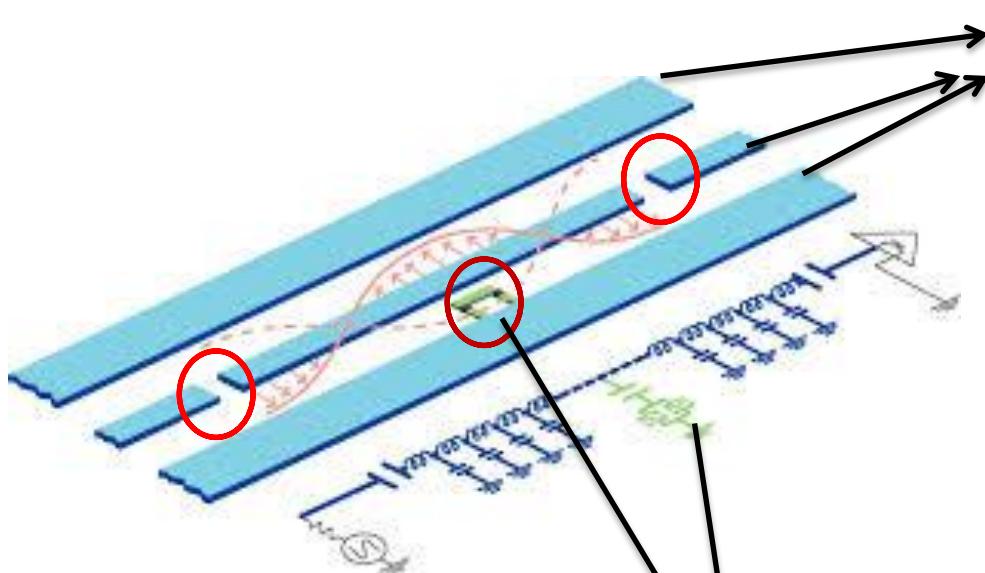
QED: Quantum Electrodynamics → interaction between light and matter (atom)



→ precise control
over quantum
system

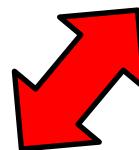
Cavity QED

CIRCUIT QED



“Light”: Transmission line of coplanar wave-guide

Mode of the field



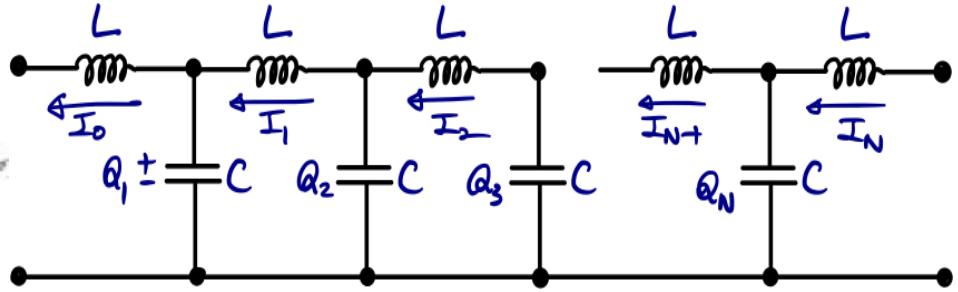
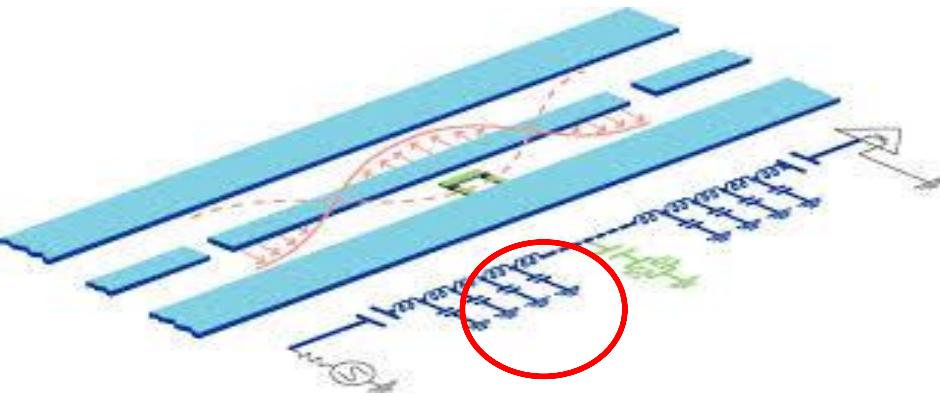
Artificial atom:
Cooper-pair box,
Josephson junction

Quantization of electromagnetic field

Basic steps:

- Finding the canonical conjugate variables q, p satisfying the PB relation $\{q_i, p_j\} = \delta_{ij}$
- Writing the Hamiltonian in quadratic form of q and $p \rightarrow$ like harmonic oscillator
- Introducing the creation and annihilation operators \rightarrow rewriting the field and Hamiltonian

Quantization of transmission line



$$L = \frac{L}{2} \sum I_j^2 - \frac{1}{2C} Q_j^2 \longrightarrow L = \frac{L}{2} \sum \left(\frac{d}{dt} \phi_j \right)^2 - \frac{1}{2C} (\phi_j - \phi_{j-1})^2$$

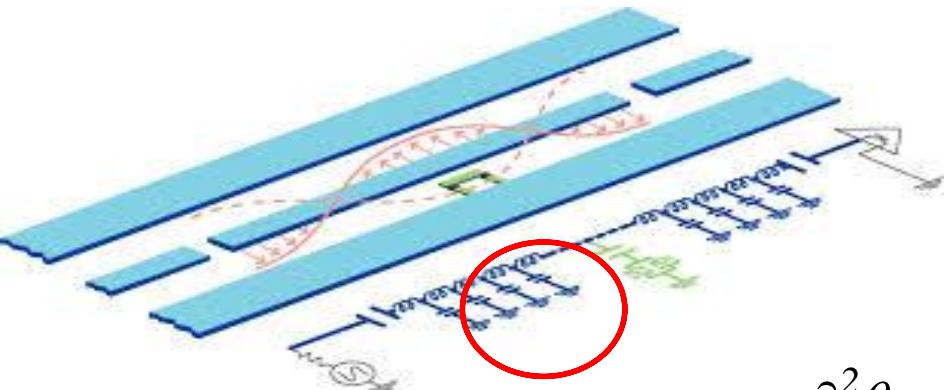
$$\phi_j = \sum_{i=1}^j Q_i$$

$$L = \int_{-L/2}^{L/2} dx \left(\frac{l}{2} j^2 - \frac{1}{2c} q^2 \right)$$

$$\theta = \int_{-L/2}^x q dx$$

$$\longrightarrow L = \int_{-L/2}^{L/2} dx \left(\frac{l}{2} \left(\frac{d}{dt} \theta \right)^2 - \frac{1}{2c} (\nabla \theta)^2 \right)$$

Quantization of transmission line



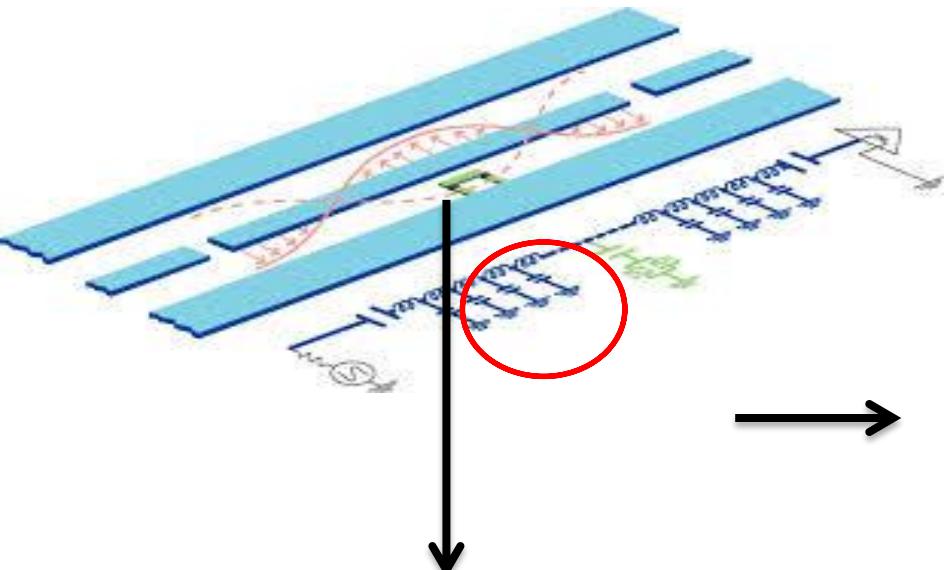
$$L = \int_{-L/2}^{L/2} dx \left(\frac{l}{2} \left(\frac{d}{dt} \theta \right)^2 - \frac{1}{2c} (\nabla \theta)^2 \right)$$

$$\rightarrow \frac{\partial^2 \theta}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 \theta}{\partial t^2} = 0$$
$$v = \sqrt{1/lc}$$

$$\rightarrow \theta(x, t) = \sqrt{\frac{2}{L}} \sum_{odd} \phi_{k_o}(t) \cos \frac{k_o \pi x}{L} + \sqrt{\frac{2}{L}} \sum_{even} \phi_{k_e}(t) \sin \frac{k_e \pi x}{L}$$

$$\rightarrow L = \sum_k \frac{l}{2} \left(\frac{d}{dt} \phi_k \right)^2 - \frac{1}{2c} \left(\frac{k \pi}{L} \right)^2 \phi_k^2$$

Quantization of transmission line



$$V(0,t) = \sqrt{\frac{\hbar\omega_2 c}{Lc}} (a_2(t) + a_2^\dagger(t))$$

$$L = \sum_k \frac{l}{2} \left(\frac{d}{dt} \phi_k \right)^2 - \frac{1}{2c} \left(\frac{k\pi}{L} \right)^2 \phi_k^2$$

$$\phi_k = \sqrt{\frac{\hbar\omega_k c}{2}} \frac{L}{k\pi} (a_k(t) + a_k^\dagger(t))$$

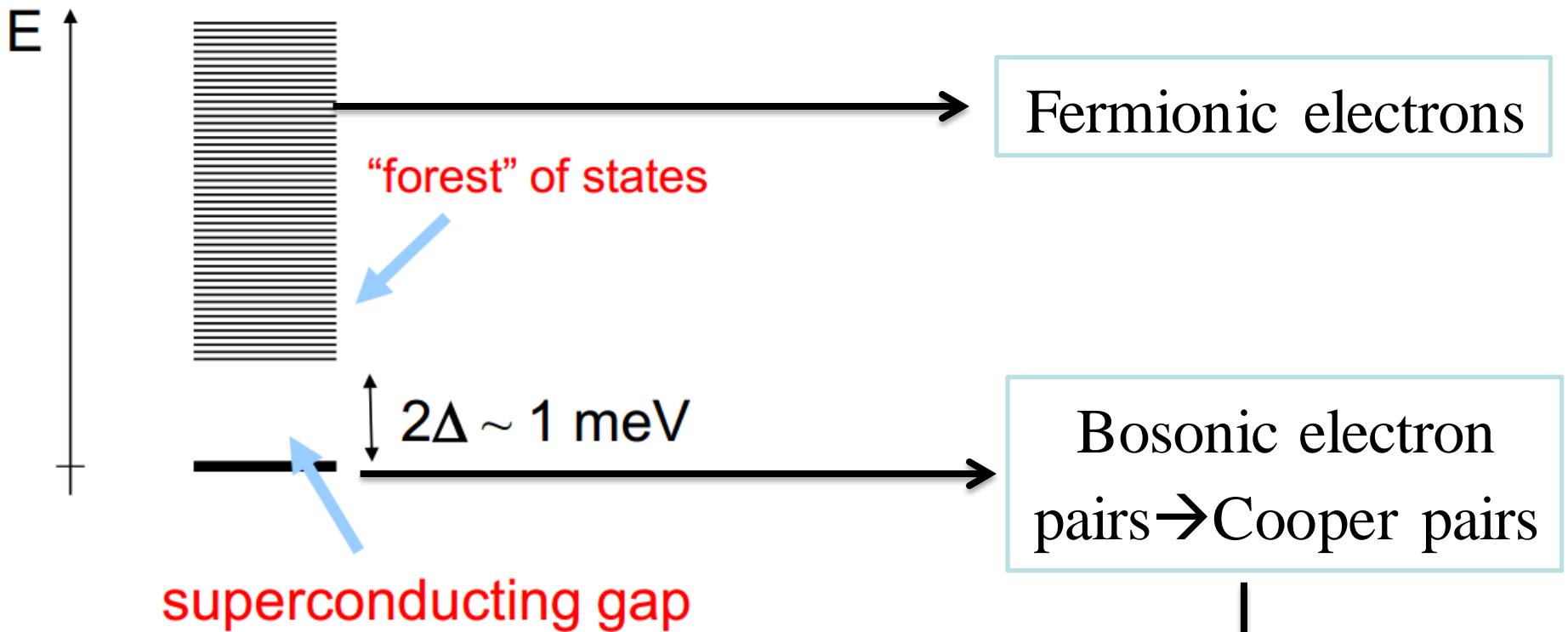
$$\pi_k = -i \sqrt{\frac{\hbar\omega_k l}{2}} (a_k(t) - a_k^\dagger(t))$$

$$\omega_k = \frac{1}{\sqrt{lc}} \frac{k\pi}{L}$$

$$V(x,t) = \frac{q}{c} = \frac{1}{c} \frac{\partial \theta}{\partial x} = - \sum_{ko} \sqrt{\frac{\hbar\omega_{ko}}{Lc}} \sin\left(\frac{ko\pi x}{L}\right) (a_{ko}(t) + a_{ko}^\dagger(t))$$

$$\rightarrow + \sum_{ke} \sqrt{\frac{\hbar\omega_{ke}}{Lc}} \cos\left(\frac{ke\pi x}{L}\right) (a_{ke}(t) + a_{ke}^\dagger(t))$$

Superconductor



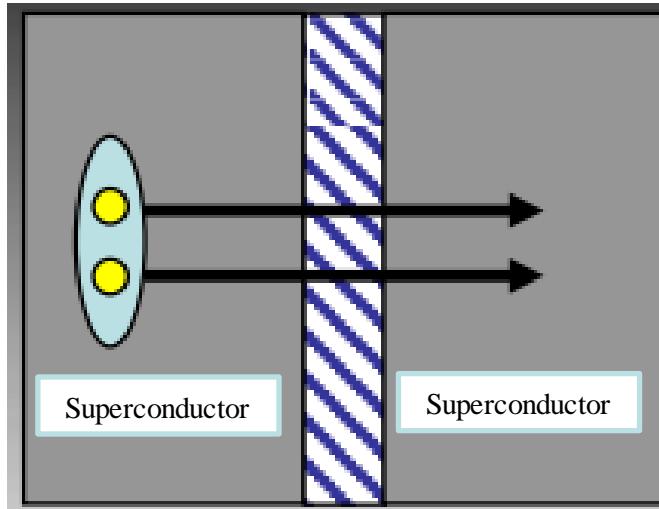
Superconductor

- ▶ dissipationless!
- ▶ provides nonlinearity via Josephson effect

$$\psi_P = \psi \exp(i \vec{P} \cdot \vec{r} / \hbar)$$

Condensate wave function

Josephson Junction


 $|n\rangle$
 $|n\rangle$


$$\Psi_P = \psi \exp\left(i \vec{P} \cdot \vec{r} / \hbar\right)$$

Number density

Phase

$$\hat{n} |n\rangle = n |n\rangle$$

$$e^{i\hat{\phi}} |\phi\rangle = e^{i\phi} |\phi\rangle$$

$$[\hat{\phi}, \hat{n}] = i$$

$$H_J \sim \sum |n\rangle \langle n+1|$$

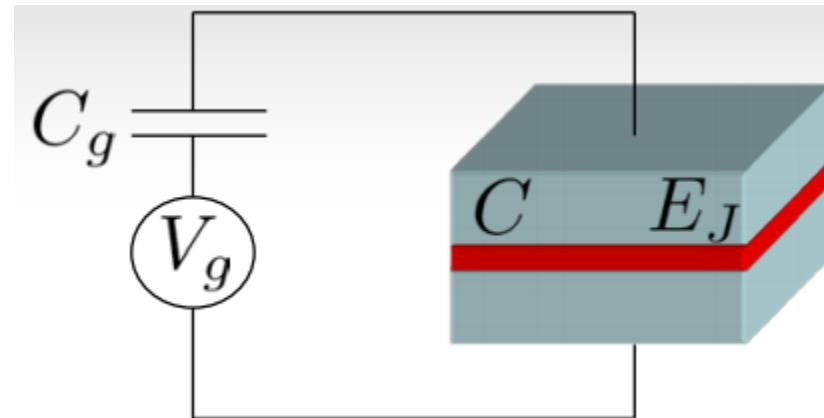
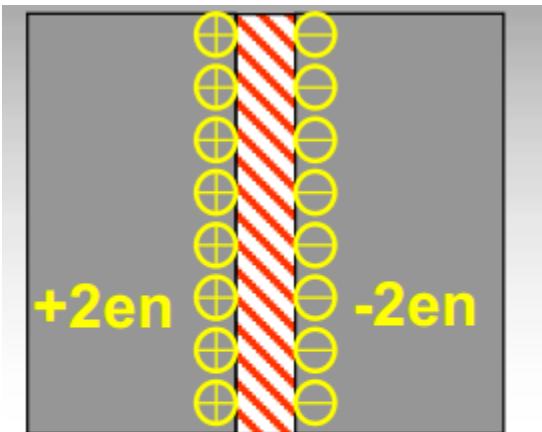
$$H_J = -\frac{E_J}{2} \sum_n |n\rangle \langle n+1| + |n+1\rangle \langle n|$$

$$e^{i\hat{\phi}} |n\rangle = |n-1\rangle$$

$$H_J = -E_J \cos \hat{\phi}$$



Cooper pair box



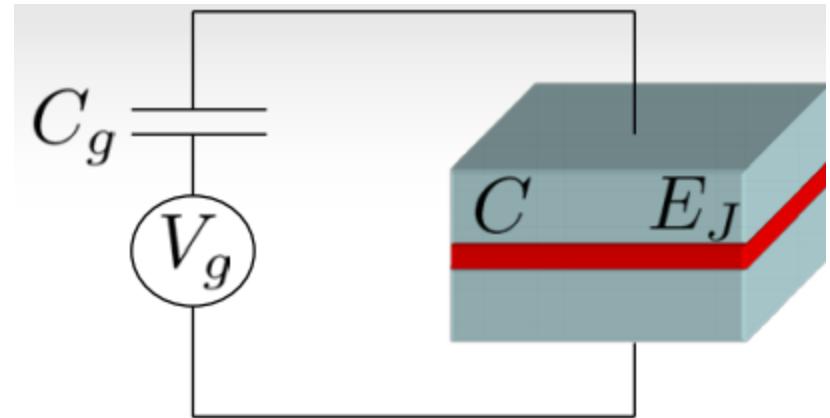
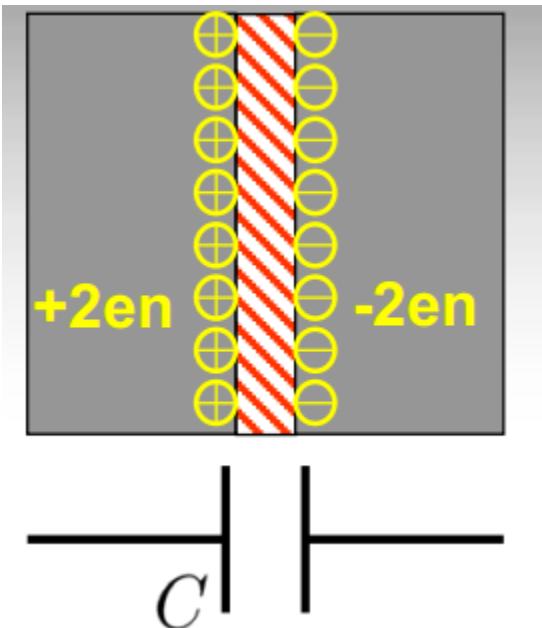
CPB: small island of superconducting material \rightarrow ground via JJ

$$H = 4E_C \sum_n \left(n - n_g \right)^2 / n \rangle \langle n | - \frac{E_J}{2} \sum_n / n \rangle \langle n+1 | + / n+1 \rangle \langle n |$$

$$E_C = \frac{e^2}{2C_{\Sigma}} \quad n_g = \frac{C_g V_g}{2e}$$

Offset charge (gate offset)

Cooper pair box



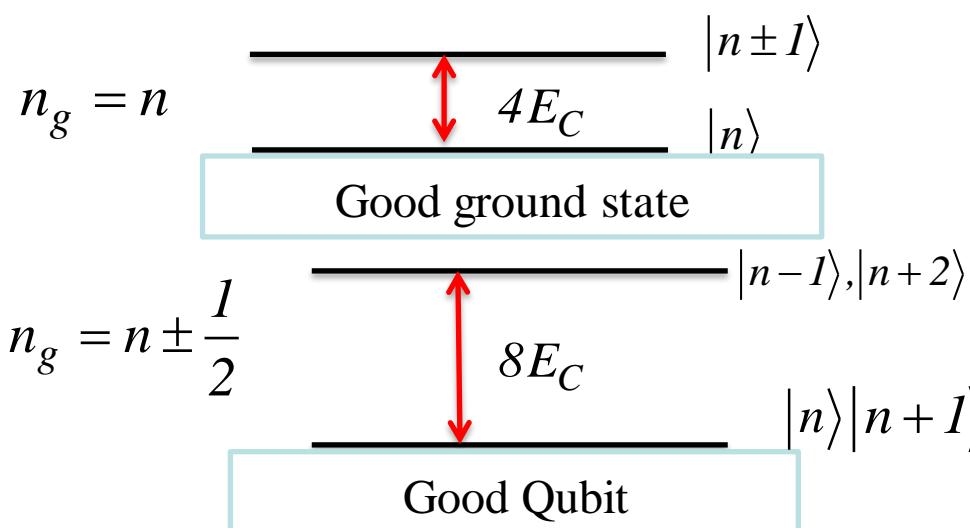
$$H = 4E_C \sum_n (n - n_g)^2 / n \rangle \langle n |$$

$$n_g = n \rightarrow E_n = 0$$

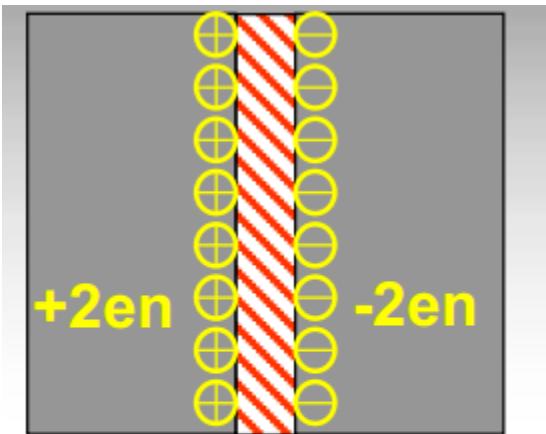
$$n_g = n \pm \frac{1}{2} \rightarrow E_n = E_C$$

$$n_g = n \pm 1 \rightarrow E_n = 4E_C$$

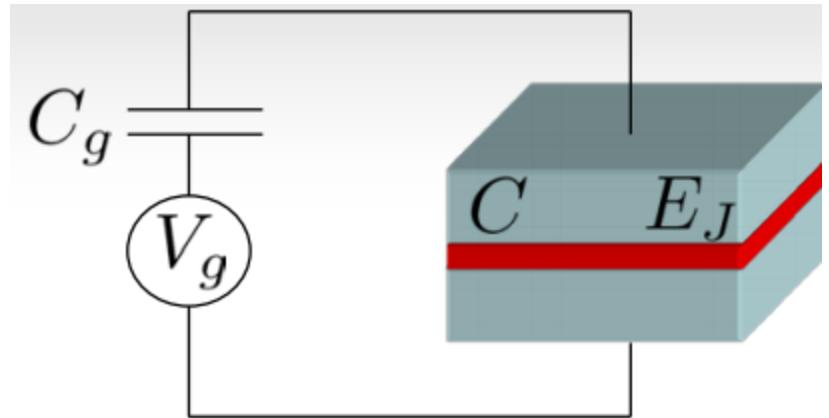
$$n_g = n \pm \frac{3}{2} \rightarrow E_n = 9E_C$$



Cooper pair box



$$n_g \rightarrow n + n_g + \frac{1}{2} \quad 4E_C \gg E_J \quad n_g \in [0,1], n=0$$

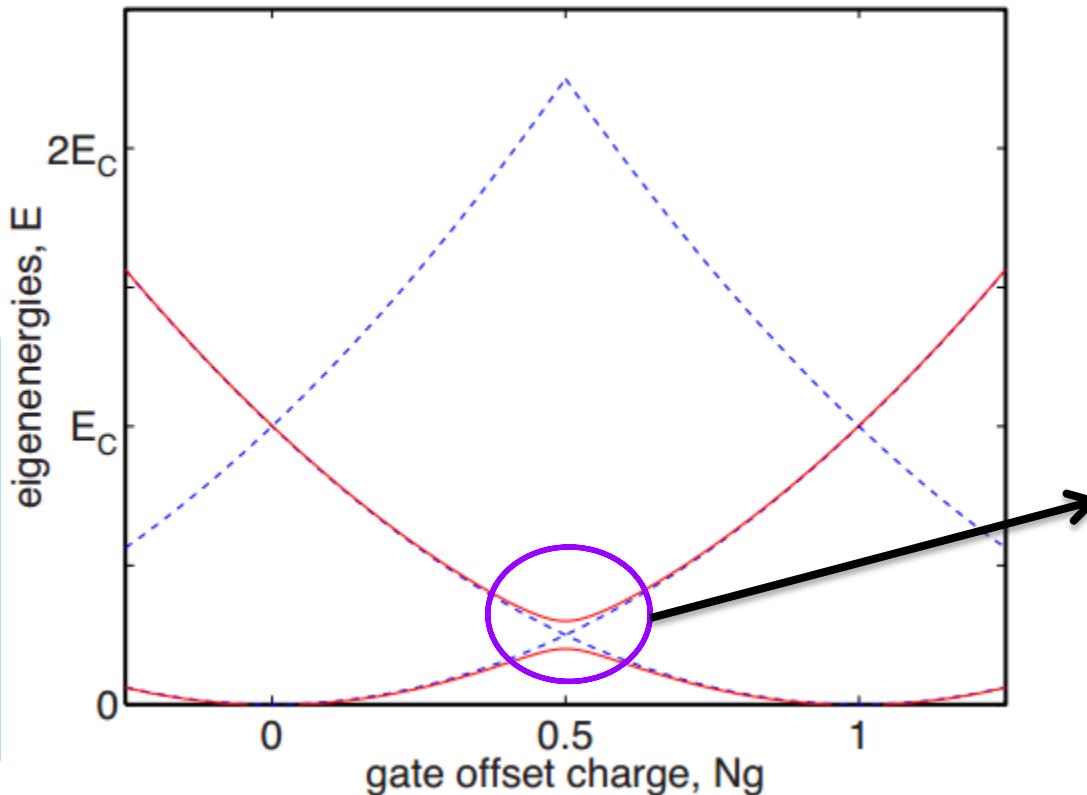


$$H_Q = 2E_C \left(N_g^2 + 1 \right) (|0\rangle\langle 0| + |1\rangle\langle 1|) - 2E_C \left(1 - 2N_g \right) (|0\rangle\langle 0| - |1\rangle\langle 1|) - \frac{E_J}{2} (|0\rangle\langle 1| + |1\rangle\langle 0|)$$

$$H_Q = -2E_C \left(1 - 2N_g \right) \sigma_z - \frac{E_J}{2} \sigma_x = \frac{E_{Ch}}{2} \sigma_z - \frac{E_J}{2} \sigma_x$$

Cooper pair box

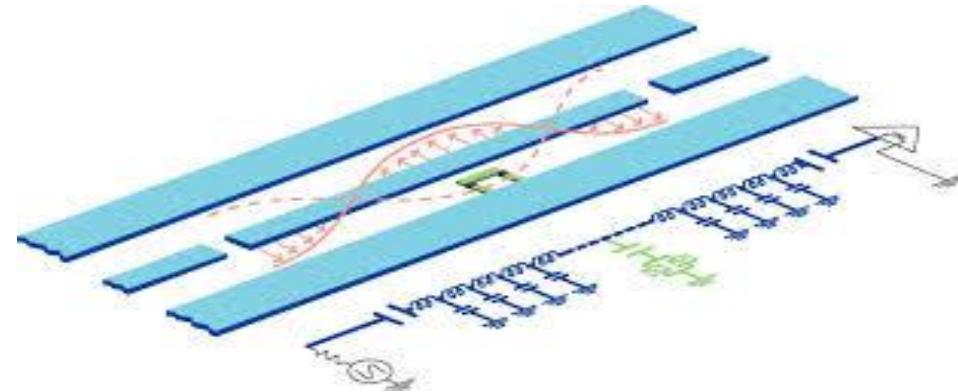
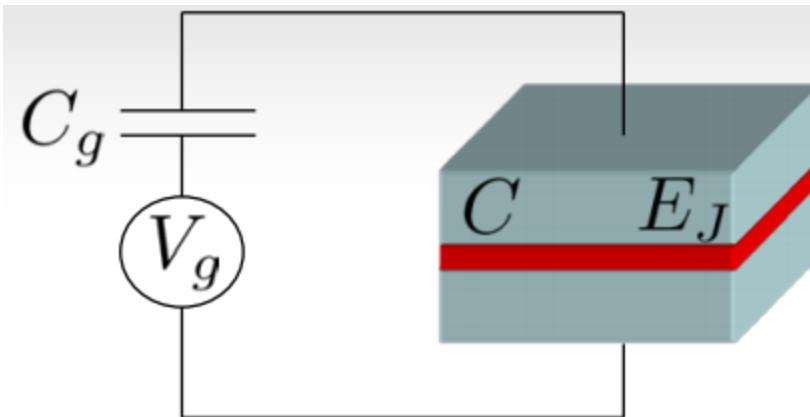
$$H_Q = -2E_C(1 - 2N_g)\sigma_z - \frac{E_J}{2}\sigma_x = \frac{E_{Ch}}{2}\sigma_z - \frac{E_J}{2}\sigma_x$$



Sweet spots:
region where
energy gap is
quadratic in
fluctuation of
 n_g

Avoided
crossing
region($E_J=0$)

Resonator + Cooper pair box



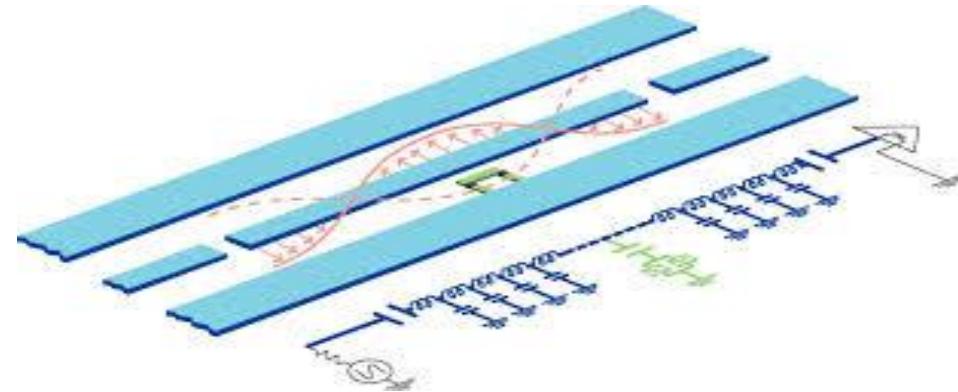
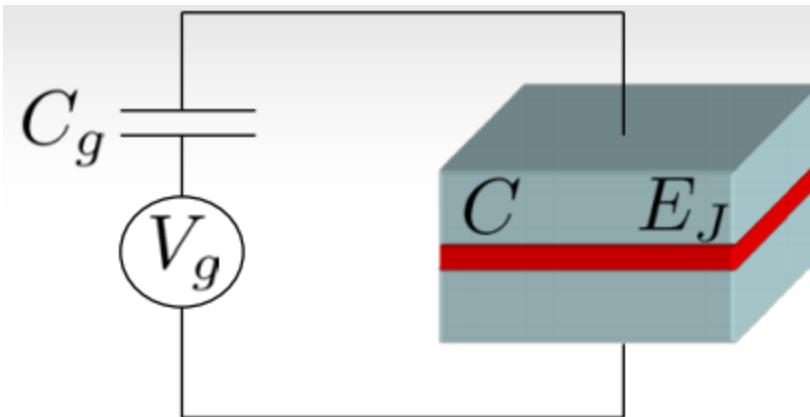
$$n_g = \frac{C_g V_g}{2e} \longrightarrow n_g = \frac{C_g (V_g + V)}{2e} = n_g^{dc} + \frac{C_g V}{2e}$$

$$V = \sqrt{\frac{\hbar\omega_r c}{Lc}} (a + a^\dagger)$$

$$\longrightarrow H = 4E_C \sum_n \left(n - n_g^{dc} - \frac{C_g V}{2e} \right)^2 |n\rangle\langle n| - \frac{E_J}{2} \sum_n |n\rangle\langle n+1| + |n+1\rangle\langle n|$$

$$\longrightarrow H_Q = \frac{E_{Ch}}{2} \sigma_z - \frac{E_J}{2} \sigma_x - e \frac{C_g}{C_\Sigma} \sqrt{\frac{\hbar\omega_r}{Lc}} (a + a^\dagger) \left(1 - 2n_g^{dc} - \sigma_z \right)$$

Resonator + Cooper pair box



$$H_Q = -\frac{E_{Ch}}{2} \sigma_z - \frac{E_J}{2} \sigma_x - e \frac{C_g}{C_\Sigma} \sqrt{\frac{\hbar \omega_r}{Lc}} (a + a^\dagger) (1 - 2n_g^{dc} - \sigma_z)$$

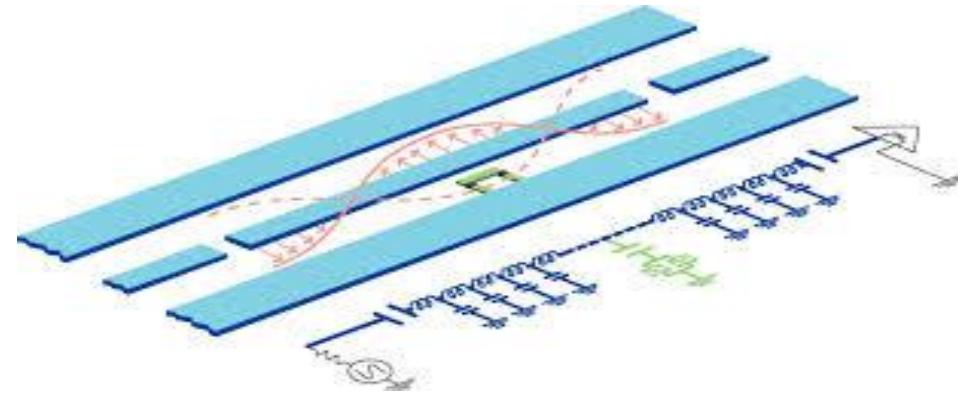
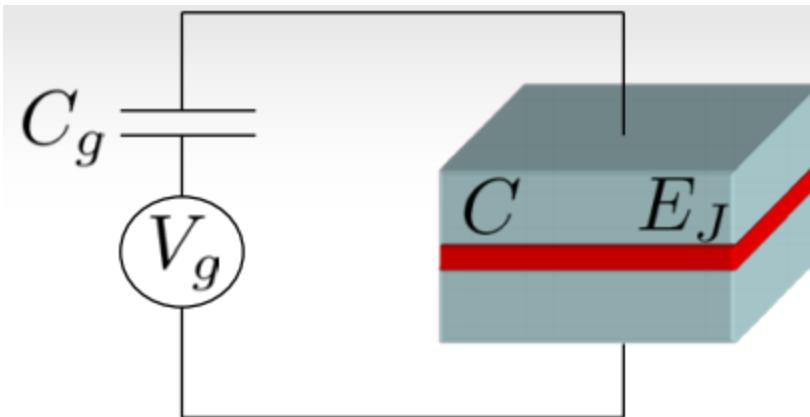
Diagonalize → new basis $|\uparrow\rangle, |\downarrow\rangle$

$$\theta = \arctan \frac{E_J}{E_{ch}}$$

$$H_{Q+R} = \hbar \omega_r \left(a^\dagger a + \frac{1}{2} \right) + \frac{\hbar \Omega}{2} \overline{\sigma_z} - e \frac{C_g}{C_\Sigma} \sqrt{\frac{\hbar \omega_r}{Lc}} (a + a^\dagger) (1 - 2n_g^{dc} - \cos(\theta) \overline{\sigma_z} + \sin(\theta) \overline{\sigma_x})$$

$$\Omega = \sqrt{{E_{Ch}}^2 + {E_J}^2} / \hbar$$

Resonator + Cooper pair box



$$H_{Q+R} = \hbar\omega_r \left(a^\dagger a + \frac{1}{2} \right) + \frac{\hbar\Omega}{2} \overline{\sigma_z} - e \frac{C_g}{C_\Sigma} \sqrt{\frac{\hbar\omega_r}{Lc}} (a + a^\dagger) \left(1 - 2n_g^{dc} - \cos(\theta) \overline{\sigma_z} + \sin(\theta) \overline{\sigma_x} \right)$$

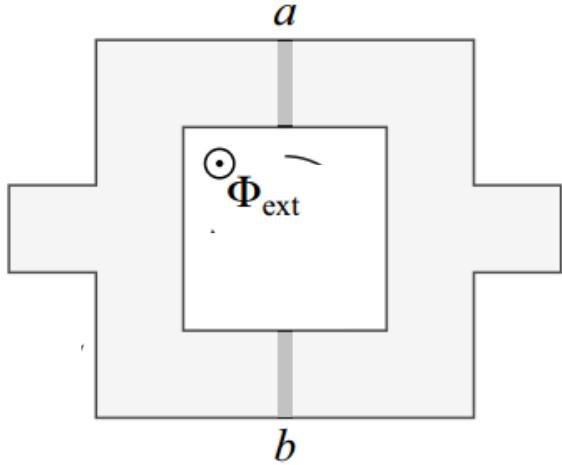
$$n_g^{dc} = \frac{1}{2}, \overline{\sigma_x} = \frac{1}{2} (\overline{\sigma^+} + \overline{\sigma^-})$$

$$\rightarrow H_{Q+R} = \hbar\omega_r \left(a^\dagger a + \frac{1}{2} \right) + \frac{\hbar\Omega}{2} \overline{\sigma_z} - \hbar g (a \overline{\sigma^+} + a^\dagger \overline{\sigma^-})$$

$$\rightarrow g = \frac{e}{2\hbar} \frac{C_g}{C_\Sigma} \sqrt{\frac{\hbar\omega_r}{Lc}}$$

(Applying
rotating wave
approximation)

Tunable Josephson junction



$$H_J = -\frac{E_J}{2} \cos \varphi_a - \frac{E_J}{2} \cos \varphi_b$$

$$\varphi_a - \varphi_b = 2\pi n + \Delta\varphi_{flux} = 2\pi n + 2\pi \frac{\Phi_{ext}}{\Phi_0}, \Phi_0 = \frac{h}{2e}$$

$$\varphi = (\varphi_a + \varphi_b) / 2 \rightarrow H_J = -E_J \cos \left(\pi \frac{\Phi_{ext}}{\Phi_0} \right) \cos \varphi$$

Effective Josephson junction energy

Resonator + Cooper pair box

$$H_{Q+R} = \hbar\omega_r \left(a^\dagger a + \frac{1}{2} \right) + \frac{\hbar\Omega}{2} \overline{\sigma_z} - \hbar g \left(a \overline{\sigma^+} + a^\dagger \overline{\sigma^-} \right)$$

$$\left| \uparrow, n+1 \right\rangle, \left| \downarrow, n \right\rangle \xrightarrow{\quad} \begin{pmatrix} \hbar\omega_r(n+1/2) & \hbar g \sqrt{n+1} \\ \hbar g \sqrt{n+1} & \hbar\omega_r(n+1/2) \end{pmatrix} \quad \begin{aligned} \left| \overline{+,n} \right\rangle &= \cos \theta_n \left| \downarrow, n \right\rangle + \sin \theta_n \left| \uparrow, n+1 \right\rangle, \\ \left| \overline{-,n} \right\rangle &= -\sin \theta_n \left| \downarrow, n \right\rangle + \cos \theta_n \left| \uparrow, n+1 \right\rangle, \end{aligned}$$

$$E_{\pm,n} = (n+1)\hbar\omega_r \pm \frac{\hbar}{2} \sqrt{4g^2(n+1) + \Delta^2} \quad \theta_n = \frac{1}{2} \tan^{-1} \left(\frac{2g\sqrt{n+1}}{\Delta} \right)$$

$$\Delta=0, n=0, \left| \overline{\pm,0} \right\rangle = \frac{1}{\sqrt{2}} \left(\left| \uparrow,1 \right\rangle \pm \left| \downarrow,0 \right\rangle \right)$$

$$\left| \downarrow,0 \right\rangle = \frac{1}{\sqrt{2}} \left(\left| \overline{+,0} \right\rangle - \left| \overline{-,0} \right\rangle \right)$$

$$\left| \uparrow,1 \right\rangle = \frac{1}{\sqrt{2}} \left(\left| \overline{+,0} \right\rangle + \left| \overline{-,0} \right\rangle \right)$$

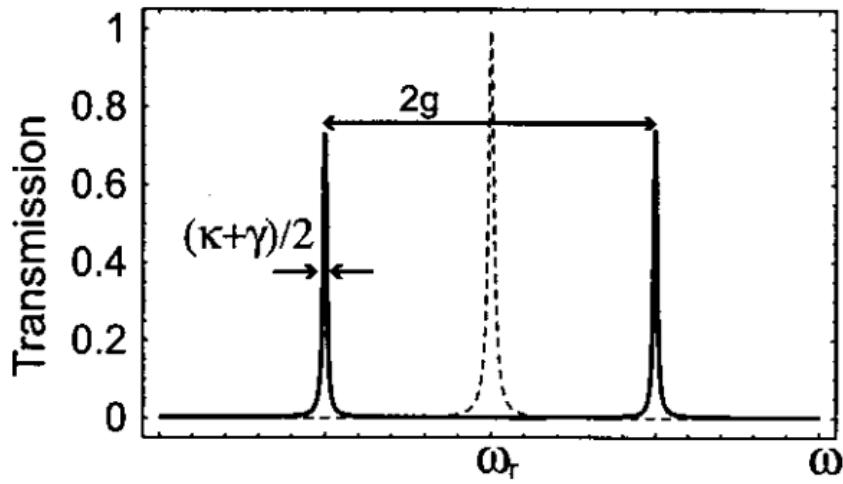
$$\left| \uparrow,1 \right\rangle = \frac{1}{\sqrt{2}} \left(\left| \overline{+,0} \right\rangle + \left| \overline{-,0} \right\rangle \right) \rightarrow \frac{1}{\sqrt{2}} \left(e^{-iE_+ t} \left| \overline{+,0} \right\rangle + e^{-iE_- t} \left| \overline{-,0} \right\rangle \right) \rightarrow \frac{1}{\sqrt{2}} e^{-i\phi} \left(\left| \overline{+,0} \right\rangle + e^{i2gt} \left| \overline{-,0} \right\rangle \right)$$

$$\left| \uparrow,1 \right\rangle \rightarrow \left| \downarrow,0 \right\rangle \rightarrow \left| \uparrow,1 \right\rangle \quad 2gt = 2\pi \rightarrow t = \frac{\pi}{g} \rightarrow f_{Rabi} = \frac{g}{\pi}$$

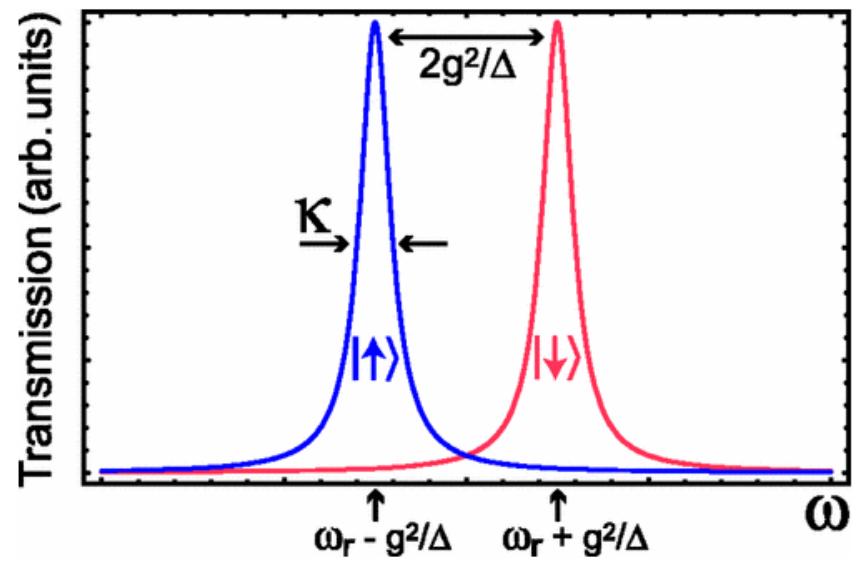
Measurement

$$E_{\pm,n} = (n+1)\hbar\omega_r \pm \frac{\hbar}{2}\sqrt{4g^2(n+1) + \Delta^2}$$

$$g / \Delta \ll 1 \quad UHU^\dagger \approx \hbar \left[\omega_r + \frac{g^2}{\Delta} \sigma_z \right] a^\dagger a + \frac{\hbar}{2} \left[\Omega + \frac{g^2}{\Delta} \right] \sigma_z$$

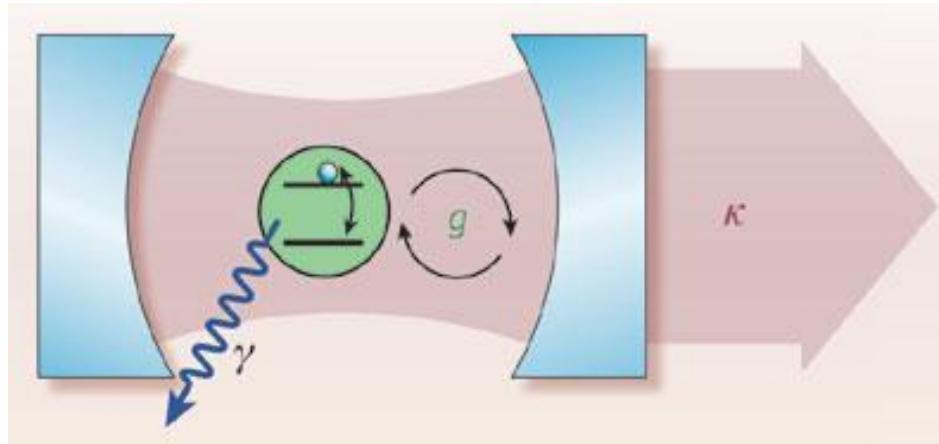


Expected measurement for zero detuning
and $n=1$



Measurement in the dispersive regime

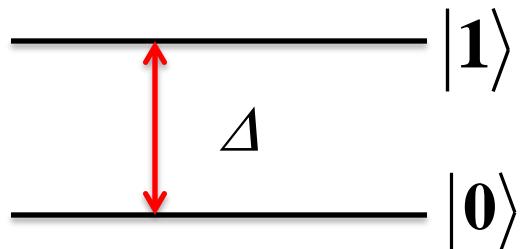
Decoherence



$$H_{Q+R} = \hbar\omega_r \left(a^\dagger a + \frac{1}{2} \right) + \frac{\hbar\Omega}{2} \overline{\sigma_z} - \hbar g \left(a \overline{\sigma^+} + a^\dagger \overline{\sigma^-} \right) + H_\kappa + H_\gamma$$

Coupling of the resonator, atom to the environment (bath)

Decoherence



$$H = -\frac{1}{2}\hbar\Delta\sigma_z$$

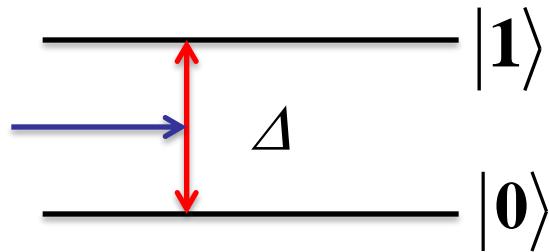
$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow[\substack{H = -\frac{1}{2}\hbar\Delta\sigma_z \\ \text{Free evolution}}]{} |\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{\frac{1}{2}i\Delta t}|0\rangle + e^{-\frac{1}{2}i\Delta t}|1\rangle\right)$$

Stationary states



$$p_0(t) = p_1(t) = \frac{1}{2} \forall t$$

How can we know it was in the superposition state?



$$p_0(t), p_1(t)$$

Oscillating

Quantum information

Two level system

$$\sigma_x \equiv X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y \equiv Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z \equiv Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$|0\rangle$ and $|1\rangle$

$$\sigma_x|0\rangle = |1\rangle, \quad \sigma_x|1\rangle = |0\rangle, \quad \sigma_x|\pm\rangle = |\pm\rangle \quad \& \quad \sigma_x|\pm i\rangle = |\mp i\rangle$$

$$|\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2}$$

$$|\pm i\rangle \equiv (|0\rangle \pm i|1\rangle)/\sqrt{2}$$

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) = \frac{1}{\sqrt{2i}}(|+i\rangle + |-i\rangle)$$

$$|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) = \frac{1}{\sqrt{2i}}(-|i\rangle - |-i\rangle)$$

Superposition states

Suppose Alice want to send Bob information by using two level system states → Bob can know by measure it in the right basis

		Bob					
		$\langle 0 $	$\langle 1 $	$\langle + $	$\langle - $	$\langle +i $	$\langle -i $
Alice	$ 0\rangle$	100%	0%	50%	50%	50%	50%
	$ 1\rangle$	0%	100%	50%	50%	50%	50%

Other bases: Bob cannot know which state Alice sent

Correlated measurements

Mixed state

$|0\rangle$ and $|1\rangle$

Pure states

$$\sigma_x \equiv X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y \equiv Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z \equiv Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\sigma_x|0\rangle = |1\rangle, \quad \sigma_x|1\rangle = |0\rangle, \quad \sigma_x|\pm\rangle = |\pm\rangle \quad \& \quad \sigma_x|\pm i\rangle = |\mp i\rangle$$

$$|\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2}$$

$$|\pm i\rangle \equiv (|0\rangle \pm i|1\rangle)/\sqrt{2}$$

		Bob								
		Alice	$\langle 0 $	$\langle 1 $	$\langle + $	$\langle - $	$\langle +i $	$\langle -i $		
ensemble of $ 0\rangle$	$ 0\rangle$	100%	0%	50%	50%	50%	50%			
	$ 1\rangle$	0%	100%	50%	50%	50%	50%			

Mixed states: cannot write in a superposition form, or single wave vector

All uncorrelated measurements \rightarrow Bob cannot know the information in any basis even if Alice tells him which basis she used

Entanglement

Combined two-level systems quantum states

$$|\mathbf{0}, \mathbf{0}\rangle \text{ or } |\mathbf{1}, \mathbf{1}\rangle \quad |\mathbf{0}, \mathbf{0}\rangle = |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle$$
$$|\mathbf{1}, \mathbf{1}\rangle = |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle$$

Suppose Xavier prepare the above quantum states. Alice and Bob want to know what it is by measure each component separately.

Alice	$\langle 0 $	$\langle 1 $	$\langle + $	$\langle - $	$\langle +i $	$\langle -i $
$\langle 0 $	50%	0%				
$\langle 1 $	0%	50%				
$\langle + $			25%	25%		
$\langle - $			25%	25%		
$\langle +i $					25%	25%
$\langle -i $					25%	25%

$$\psi = |\psi_1\rangle \otimes |\psi_2\rangle$$

$$p_{\phi_1\phi_2} = \langle \phi_1 | \psi_1 \rangle \otimes \langle \phi_2 | \psi_2 \rangle$$

There is one basis in which measurements of Bob and Alice are correlated

Entanglement

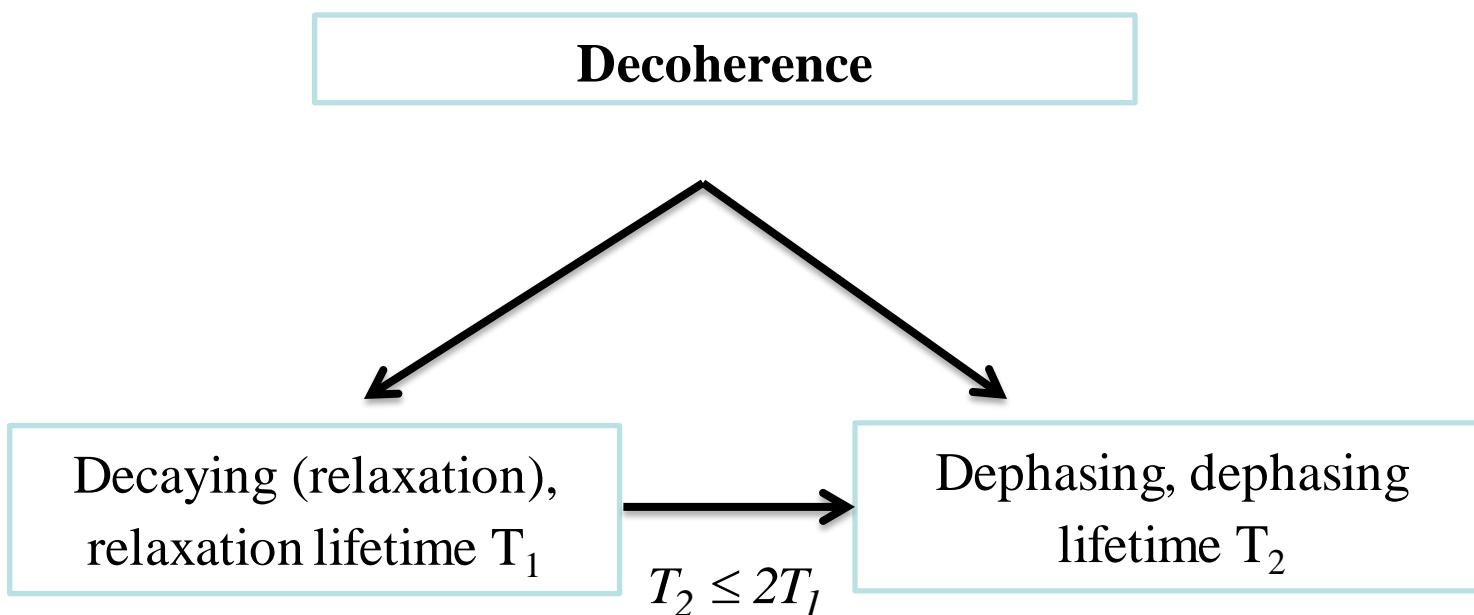
$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0,0\rangle + |1,1\rangle) \equiv |\phi^+\rangle$$

		Bob					
		$\langle 0 $	$\langle 1 $	$\langle + $	$\langle - $	$\langle +i $	$\langle -i $
Alice		50%	0%	25%	25%	25%	25%
$\langle 0 $		0%	50%	25%	25%	25%	25%
$\langle 1 $		25%	25%	50%	0%	25%	25%
$\langle + $		25%	25%	0%	50%	25%	25%
$\langle - $		25%	25%	0%	50%	25%	25%
$\langle +i $		25%	25%	25%	25%	0%	50%
$\langle -i $		25%	25%	25%	25%	50%	0%

There are **three bases** in which measurements of Bob and Alice are correlated

→ **Entanglement states**

Decoherence

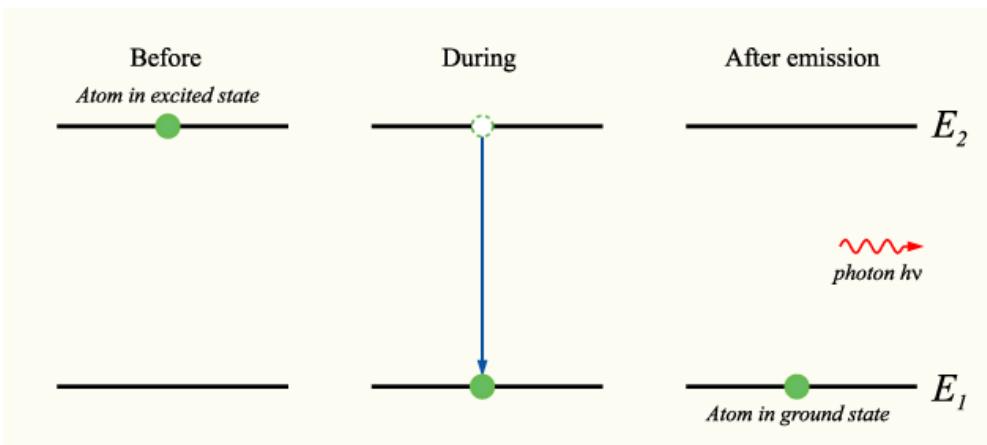


Decaying

Losing energy to the environment

Probability finding the system in excited state

$$p_1(t) = e^{-\gamma t} \rightarrow T_1 = \frac{1}{\gamma}$$



Dephasing

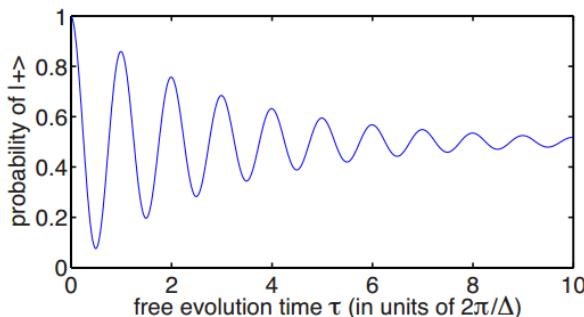
$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow[H = -\frac{I}{2}\hbar\Delta\sigma_z]{} |\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{\frac{I}{2}i\Delta t}|0\rangle + e^{-\frac{I}{2}i\Delta t}|1\rangle\right)$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$p_+(t) = \langle + | \psi(t) \rangle = \frac{1}{2} + \frac{1}{2} \cos(\Delta t)$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{-i\phi}|1\rangle) \xrightarrow[+]{} p_+(t) = \langle + | \psi(t) \rangle = \frac{1}{2} + \frac{1}{2} \cos(\Delta t - \phi)$$

Random phase kick
from environment



Average measurements

$$p_+(t) = \frac{1}{2} + \frac{1}{2} \cos(\Delta t - \phi/2) \cos(\phi/2)$$

Reduced amplitude

Treatment of decoherence

Density matrix

Pure state

$$\rho = |\chi\rangle\langle\chi|$$

Mix state

$$\rho = W_a |\chi_a\rangle\langle\chi_a| + W_b |\chi_b\rangle\langle\chi_b| + \dots$$

Pure state 2-level system

$$|\chi\rangle = a_1 |0\rangle + a_2 |1\rangle$$

$$\rho = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} a_1^* & a_2^* \\ a_2^* & a_1^* \end{pmatrix} = \begin{pmatrix} |a_1|^2 & a_1 a_2^* \\ a_1^* a_2 & |a_2|^2 \end{pmatrix}$$

$$Tr(\rho) = 1$$

$$\langle A \rangle = Tr(\rho A)$$



Diagonal elements: probability to find the system in the pure state.

Relaxation: diagonal element decaying

Treatment of decoherence

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \xrightarrow[H = -\frac{1}{2}\hbar\Delta\sigma_z]{} |\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{\frac{1}{2}i\Delta t}|0\rangle + e^{-\frac{1}{2}i\Delta t}|1\rangle\right)$$

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| = \frac{1}{2}\begin{pmatrix} 1 & e^{i\Delta t} \\ e^{-i\Delta t} & 1 \end{pmatrix}$$

Strongly depends of the basis representation!!!

$$p_+(t) = \langle + | \rho(t) | + \rangle = \frac{1}{4}(1 \quad 1)\begin{pmatrix} 1 & e^{i\Delta t} \\ e^{-i\Delta t} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} + \frac{1}{2}\cos(\Delta t)$$

→ Oscillation (intference) term coming from the off-diagonal elements
 → coherent superposition

$$\rho(t) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Mixed state → no coherent element in this basis

Master equation

Schrodinger equation

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = H(t)|\psi(t)\rangle$$

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle$$

$$i\hbar \frac{\partial |U(t)\rangle}{\partial t}|\psi(0)\rangle = H(t)U(t)|\psi(0)\rangle$$

$$i\hbar \frac{\partial U(t)}{\partial t} = H(t)U(t)$$

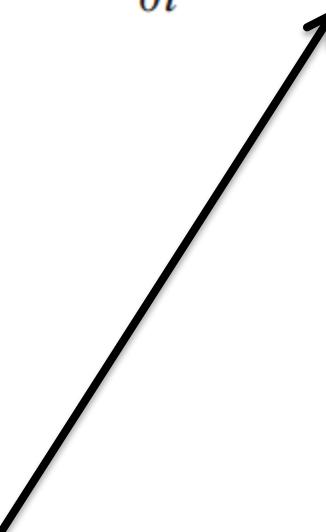
$$\rho(t) = \sum_n W_n |\psi(t)_n\rangle \langle \psi(t)_n|$$

$$= \sum_n W_n U(t) |\psi(0)_n\rangle \langle \psi(0)_n| U(t)^\dagger$$

$$\begin{aligned} i\hbar \frac{\partial \rho(t)}{\partial t} &= i\hbar \frac{\partial U(t)}{\partial t} \rho(0) U(t)^\dagger + i\hbar U(t) \rho(0) \frac{\partial U^\dagger}{\partial t} \\ &= H(t) U(t) \rho(0) U(t)^\dagger - U(t) \rho(0) U(t)^\dagger H(t) \end{aligned}$$

Liouville equation

$$i\hbar \frac{\partial \rho(t)}{\partial t} = [H(t), \rho(t)]$$



Master equation

Interaction picture

$$H(t) = H_0 + V(t)$$

$$|\psi(t)\rangle = e^{-(i/\hbar)H_0t} |\psi(t)_I\rangle$$

$$|\psi(t)_I\rangle = e^{(i/\hbar)H_0t} U(t) |\psi(0)\rangle$$

$$V(t)_I = e^{(i/\hbar)H_0t} V(t) e^{-(i/\hbar)H_0t}$$

$$\rho(t) = (e^{-(i/\hbar)H_0t}) \rho(t)_I e^{+(i/\hbar)H_0t}$$

$$\dot{\rho}(t)_I = -(i/\hbar)[V(t)_I, \rho(0)] - (i/\hbar)^2 \int_0^t dt' [V(t)_I, [V(t')_I, \rho(t')_I]]$$

Liouville equation

$$i\hbar \frac{\partial \rho(t)_I}{\partial t} = [V(t)_I, \rho(t)_I]$$

$$\rho(t)_I = \rho(0)_I - \frac{i}{\hbar} \int_0^t [V(\tau)_I, \rho(\tau)_I] d\tau$$

$$\rho(t)_I = \rho(0)_I - \frac{i}{\hbar} \int_0^t [V(\tau)_I, \rho(0)_I] d\tau$$



Master equation

$$H_{tot} = H_S + H_R + H_{SR}(t)$$

$$H_{SR}(t) = V(t) \quad \rho(t) = \underset{R}{Tr} \rho_{tot}(t)$$

$$\dot{\rho}_I(t) = \frac{-i}{\hbar} \underset{R}{Tr} [H_{SR,I}(t), \rho_{tot,I}(0)] - \left(\frac{i}{\hbar} \right)^2 \int_0^t dt' \underset{R}{Tr} [H_{SR,I}(t), [H_{SR,I}(t'), \rho_{tot,I}(t')]]$$

$$\rho_{tot}(0) = \rho(0) \otimes \rho_R$$

Suppose H_{SR} has no diagonal elements in the bath basis

$$\underset{R}{Tr} [H_{SR,I}(t), \rho_{tot,I}(0)] = \rho_I(0) \underset{R}{Tr} [H_{SR,I}(t), \rho_R] = 0$$

$$\dot{\rho}_I(t) = - \left(\frac{1}{\hbar} \right)^2 \int_0^t dt' \underset{R}{Tr} [H_{SR,I}(t), [H_{SR,I}(t'), \rho_{tot,I}(t')]]$$

Master equation

$$\rho_{tot,I}(t') = \rho_I(t') \otimes \rho_R$$

Weak coupling approximation

$$\rho_I(t') \rightarrow \rho_I(t)$$

Markoff approximation

$$\dot{\rho}_I(t) = -\left(\frac{1}{\hbar}\right)^2 \int_0^t dt' \text{Tr}_R \left[H_{SR,I}(t), [H_{SR,I}(t'), \rho_I(t) \otimes \rho_R] \right]$$

$$H_{SR} = \sum_{\alpha} A_{\alpha} \otimes B_{\alpha} \quad H_{SR}(t) = \sum_{\alpha} A_{\alpha}(t) \otimes B_{\alpha}(t)$$

$$\langle Q \rangle = \text{Tr}_R(\rho Q) \quad \text{Tr}(ABC) = \text{Tr}(CAB)$$

$$\begin{aligned} \langle B_{\alpha}(t) B_{\beta}(t') \rangle &= \text{Tr}_R \left[e^{\frac{iH_R t}{\hbar}} B_{\alpha} e^{-\frac{iH_R t}{\hbar}} e^{\frac{iH_R t'}{\hbar}} B_{\beta} e^{-\frac{iH_R t}{\hbar}} \rho_R \right] = \text{Tr}_R \left[e^{\frac{iH_R(t-t')}{\hbar}} B_{\alpha} e^{-\frac{iH_R(t-t')}{\hbar}} B_{\beta} \rho_R \right] \\ &= \langle B_{\alpha}(t-t') B_{\beta}(0) \rangle \end{aligned}$$

Master equation

$$\langle B_\alpha(t) \rangle = 0 \quad t - t' \gg \tau \quad \langle B_\alpha(t) B_\beta(t') \rangle \simeq \langle B_\alpha(t) \rangle \langle B_\beta(t') \rangle = 0$$

$$t'' = t - t', dt'' = -dt'$$

$$\dot{\rho}_I(t) = -\left(\frac{1}{\hbar}\right)^2 \sum_{\alpha, \beta} \int_0^\infty dt'' \left\{ [A_\alpha(t), A_\beta(t-t'')] \rho_I(t) \right\} \langle B_\alpha(t'') B_\beta(0) \rangle - \left[A_\alpha(t), \rho_I(t) A_\beta(t-t'') \right] \langle B_\beta(0) B_\alpha(t'') \rangle$$

Superoperator

$$H_{SR} = \sum_{\alpha} A_{\alpha} \otimes B_{\alpha} = \sum_{\alpha} \sum_{\omega} A_{\alpha}(\omega) B_{\alpha}$$

$$[H_S, A_{\alpha}(\omega)] = -\omega A_{\alpha}(\omega) \quad A_{\alpha}(\omega, t) = A_{\alpha}(\omega) e^{-i\omega t}$$

$$[H_S, A_{\alpha}^{\dagger}(\omega)] = +\omega A_{\alpha}^{\dagger}(\omega) \quad H_{SR} = \sum_{\alpha} \sum_{\omega} A_{\alpha}(\omega) B_{\alpha}^{\dagger} = \sum_{\alpha} \sum_{\omega} A_{\alpha}^{\dagger}(\omega) B_{\alpha}$$

Master equation

$$\dot{\rho}_I(t) = -\left(\frac{1}{\hbar}\right)^2 \sum_{\alpha,\beta} \int_0^\infty dt'' \left\{ \left[A_\alpha(t), A_\beta(t-t'') \rho_I(t) \right] \langle B_\alpha(t'') B_\beta(0) \rangle - \left[A_\alpha(t), \rho_I(t) A_\beta(t-t'') \right] \langle B_\beta(0) B_\alpha(t'') \rangle \right\}$$

$H_{SR} = \sum_{\alpha} \sum_{\omega} A_\alpha^\dagger(\omega) B_\alpha$
 $H_{SR} = \sum_{\beta} \sum_{\omega'} A_\beta(\omega') B_\beta^\dagger$
 $H_{SR} = \sum_{\alpha} \sum_{\omega} A_\alpha(\omega) B_\alpha^\dagger$
 $H_{SR} = \sum_{\beta} \sum_{\omega'} A_\beta^\dagger(\omega') B_\beta$

$$\dot{\rho}_I(t) = \left(\frac{1}{\hbar}\right)^2 \sum_{\omega,\omega'} e^{i(\omega-\omega')t} \sum_{\alpha,\beta} \Gamma_{\alpha\beta}(\omega') \left[A_\beta(\omega') \rho_I(t) A_\alpha^\dagger(\omega) - A_\alpha^\dagger(\omega) A_\beta(\omega') \rho_I(t) \right] + h.c$$

$$\Gamma_{\alpha\beta}(\omega') = \int_0^\infty dt'' e^{i\omega' t''} \langle B_\alpha(t'') B_\beta^\dagger(0) \rangle = \int_0^\infty dt e^{i\omega' t} \langle B_\alpha(t) B_\beta^\dagger(0) \rangle$$

$$\begin{aligned} \dot{\rho}_I(t) &= \left(\frac{1}{\hbar}\right)^2 \sum_{\omega,\omega'} \sum_{\alpha,\beta} \Gamma_{\alpha\beta}(\omega') \left[A_\beta(\omega',t) \rho_I(t) A_\alpha^\dagger(\omega,t) - A_\alpha^\dagger(\omega,t) A_\beta(\omega',t) \rho_I(t) \right] \\ &\quad + \left(\frac{1}{\hbar}\right)^2 \sum_{\omega,\omega'} \sum_{\alpha,\beta} \Gamma_{\beta\alpha}^*(\omega') \left[A_\alpha(\omega,t) \rho_I(t) A_\beta^\dagger(\omega',t) - \rho_I(t) A_\beta^\dagger(\omega',t) A_\alpha(\omega',t) \right] \end{aligned}$$

Master equation

$$\dot{\rho}_I(t) = \left(\frac{1}{\hbar}\right)^2 \sum_{\omega, \omega'} \sum_{\alpha, \beta} \Gamma_{\alpha\beta}(\omega') \left[A_\beta(\omega', t) \rho_I(t) A_\alpha^\dagger(\omega, t) - A_\alpha^\dagger(\omega, t) A_\beta(\omega', t) \rho_I(t) \right]$$

$$+ \left(\frac{1}{\hbar}\right)^2 \sum_{\omega, \omega'} \sum_{\alpha, \beta} \Gamma_{\beta\alpha}^*(\omega') \left[A_\alpha(\omega, t) \rho_I(t) A_\beta^\dagger(\omega', t) - \rho_I(t) A_\beta^\dagger(\omega', t) A_\alpha(\omega, t) \right]$$

$$\Gamma_{\alpha\beta}(\omega') = \frac{1}{2} \gamma_{\alpha\beta}(\omega') + i S_{\alpha\beta}(\omega')$$

$$\Gamma_{\beta\alpha}^*(\omega') = \frac{1}{2} \gamma_{\alpha\beta}(\omega') - i S_{\alpha\beta}(\omega')$$

$$\dot{\rho}_I(t) = -i \left(\frac{1}{\hbar}\right)^2 \sum_{\omega, \omega'} \sum_{\alpha, \beta} S_{\alpha\beta}(\omega') \left[A_\alpha^\dagger(\omega, t) A_\beta(\omega', t), \rho_I(t) \right]$$

$$+ \left(\frac{1}{\hbar}\right)^2 \sum_{\omega, \omega'} \sum_{\alpha, \beta} \gamma_{\alpha\beta}(\omega') \left[A_\beta(\omega', t) \rho_I(t) A_\alpha^\dagger(\omega, t) - \frac{1}{2} \{ A_\alpha^\dagger(\omega, t) A_\beta(\omega', t), \rho_I(t) \} \right]$$

Master equation

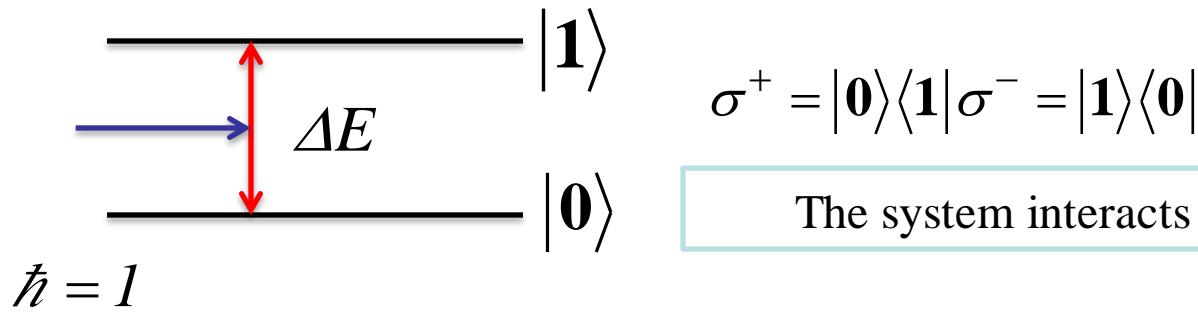
$$\begin{aligned}\dot{\rho}_I(t) = & -i\left(\frac{1}{\hbar}\right)^2 \sum_{\omega,\omega'} \sum_{\alpha,\beta} S_{\alpha\beta}(\omega') \left[A_\alpha^\dagger(\omega,t) A_\beta(\omega',t), \rho_I(t) \right] \\ & + \left(\frac{1}{\hbar}\right)^2 \sum_{\omega,\omega'} \sum_{\alpha,\beta} \gamma_{\alpha\beta}(\omega') \left[A_\beta(\omega',t) \rho_I(t) A_\alpha^\dagger(\omega,t) - \frac{1}{2} \left\{ A_\alpha^\dagger(\omega,t) A_\beta(\omega',t), \rho_I(t) \right\} \right]\end{aligned}$$

$$\omega \neq \omega' \tau_S = \frac{1}{|\omega - \omega'|} \gg \tau \rightarrow \omega \approx \omega'$$

Master equation in Schrodinger Picture

$$\begin{aligned}\dot{\rho}(t) = & -i[H_S, \rho] - i\left(\frac{1}{\hbar}\right)^2 \sum_{\omega} \sum_{\alpha,\beta} S_{\alpha\beta}(\omega) \left[A_\alpha^\dagger(\omega) A_\beta(\omega), \rho(t) \right] \\ & + \left(\frac{1}{\hbar}\right)^2 \sum_{\omega} \sum_{\alpha,\beta} \gamma_{\alpha\beta}(\omega) \left[A_\beta(\omega) \rho(t) A_\alpha^\dagger(\omega) - \frac{1}{2} \left\{ A_\alpha^\dagger(\omega) A_\beta(\omega), \rho(t) \right\} \right]\end{aligned}$$

Master equation



$$\sigma^+ = |\mathbf{0}\rangle\langle\mathbf{1}| \quad \sigma^- = |\mathbf{1}\rangle\langle\mathbf{0}|$$

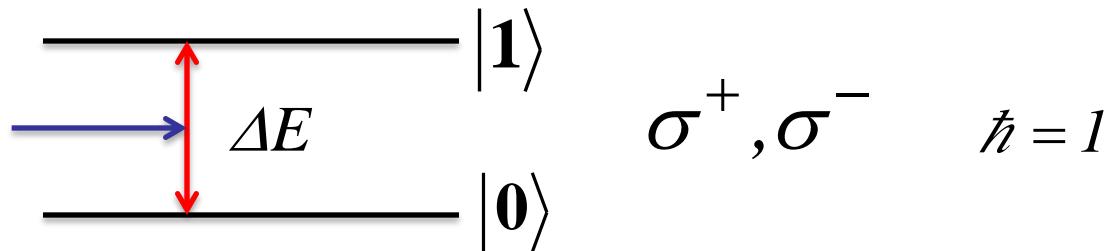
The system interacts with the environment

$$\dot{\rho}_I(t) = -iS[\sigma^+\sigma^-, \rho_I(t)] + \gamma \left[\sigma^-\rho_I(t)\sigma^+ - \frac{1}{2}(\sigma^+\sigma^-\rho_I(t) + \rho_I(t)\sigma^+\sigma^-) \right]$$

$$\sigma^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \sigma^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \dot{\rho}_{11} & \dot{\rho}_{12} \\ \dot{\rho}_{21} & \dot{\rho}_{22} \end{pmatrix} = \begin{pmatrix} -\gamma\rho_{11} & -iS\rho_{12} - \frac{1}{2}\gamma\rho_{12} \\ iS\rho_{12} - \frac{1}{2}\gamma\rho_{21} & \gamma\rho_{11} \end{pmatrix}$$

Master equation



$$\left\{ \begin{array}{l} \rho_{11} = 1 - \rho_{22}(0) e^{-\gamma t} \\ \\ \rho_{12} = \rho_{12}(0) \cdot e^{\left(iS - \frac{1}{2}\gamma\right)t} \quad T_2 \leq 2T_1 = \frac{2}{\gamma} \\ \\ \rho_{12} = \rho_{12}(0) \cdot e^{\left(-iS - \frac{1}{2}\gamma\right)t} \\ \\ \rho_{22} = \rho_{22}(0) e^{-\gamma t} \end{array} \right.$$

**THANK YOU FOR
YOUR LISTENING!**